KINETIC THEORY OF WAVE PROPAGATION IN RAREFIED GASES AND PLASMAS

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LIST OF SYMBOLS

a	speed of sound
a_o	adiabatic speed of sound, $a_o = (\gamma \frac{kT_o}{m})^{1/2}$
$\bar{a}$	acceleration vector
A_o	adiabatic speed of sound for a mixture, $A_o = (\frac{\gamma kT_o}{\alpha m + \alpha^* m^*})^{1/2}$
$\bar{\bar{A}}$	complex matrix
$A_{\ell i}$	$(\ell, i)^{th}$ element of $\bar{\bar{A}}$
$\bar{c}, \bar{c}^*$	nondimensional molecular velocity vectors, $\bar{c} = \beta_o \bar{v}$, $\bar{c}^* = \beta_o^* \bar{v}$
c_i, c_i^*	i^{th} discrete velocity point on c_x and c_x^* axis respectively
$\hat{e}$	electron charge
E	electric field
f, f^*	distribution functions
f_o	(Maxwellian distribution for molecules of mass m)/β_o^3
f_o^*	(Maxwellian distribution for molecules of mass m*)/β_o^{*3}
F, F^*, F^{**}	the forces at a distance R for m-m, m-m* and m*-m* collisions, respectively
H_i	i^{th} quadrature weighting coefficient
i	index
$\hat{i}$	the imaginary unit, $\hat{i} = (-1)^{1/2}$

LIST OF SYMBOLS (Continued)

$\bar{\bar{I}}$	identity matrix
j	index
k	Boltzmann's constant
K	wave number, $K = K_R + \hat{i}K_I$
$\hat{K}$	nondimensional wave number, $\hat{K} = \frac{Ka_o}{\omega}$
$\hat{K}_M$	nondimensional wave number, $\hat{K}_M = \frac{KA_o}{\omega_{Mo}}$
$\hat{K}_p$	nondimensional wave number, $\hat{K}_p = \frac{K(\frac{kT_o}{m})^{1/2}}{\omega_p}$
K_n	Knudsen number
ℓ	index, mean free path
$\mathcal{L}(\ )$	Defined in Chapter V
m, m^*	molecular masses
$\hat{m}, \hat{m}^*$	reduced masses, $\hat{m} = \frac{m}{m + m^*}$, $\hat{m}^* = \frac{m^*}{m + m^*}$
m_o	$m_o = m + m^*$
M	number of sub-intervals in interval $o \leq \xi \leq 1$
n	number of positive discrete velocity points
$\hat{n}$	index
n', n'^*	number density perturbations
n_o, n_o^*	equilibrium number densities
p_o, p_o^*	equilibrium pressures (partial pressures in Chapters IV and V)
p'_{ij}	perturbation pressure tensor

LIST OF SYMBOLS (Continued)

Pr	Prandtl number
$\bar{q}'$, $\bar{q}'^*$	macroscopic velocity vectors
r	$r = p_o/\mu\omega$ for forced sound propagation $r = p_o/\mu\omega_o$ for free sound propagation
r_M	$r_M = p_o/\mu\omega$ for forced sound propagation in a mixture (p_o is a partial pressure) $r_M = p_o/\mu\omega_{Mo}$ for free sound propagation in a mixture (p_o is a partial pressure)
r_M^*	$r_M^* = p_o^*/\mu^*\omega$ for forced sound propagation in a mixture (p_o^* is a partial pressure) $r_M^* = p_o^*/\mu^*\omega_{Mo}$ for free sound propagation in a mixture (p_o^* is a partial pressure)
R	distance between two molecules
S	parameter indicative of spacing between transmitter and receiver in highly rarefied gas, $S = \sqrt{2}\,\omega\beta_o x$
t	time
T_o	equilibrium temperature
T', T'^*	perturbation temperatures
u_o	nondimensional piston velocity
$\bar{v}$	microscopic velocity vector with components v_x, v_y, v_z
x,y,z	physical space coordinates
α	number density fraction of m - molecules, $\alpha = n_o/(n_o + n_o^*)$
α^*	number density fraction of m^* - molecules, $\alpha^* = n_o^*/(n_o + n_o^*)$

LIST OF SYMBOLS (Continued)

β_o	(most probable speed of molecules of mass m)$^{-1}$, $\beta_o = (\frac{m}{2kT_o})^{\frac{1}{2}}$
β_o^*	(most probable speed of molecules of mass m*)$^{-1}$, $\beta_o^* = (\frac{m^*}{2kT_o})^{1/2}$
γ	ratio of specific heats
$\delta_{\ell i}$	Kronecker delta
ε	parameter used in coordinate transformation
ε^*	$\varepsilon^* = \Theta\varepsilon$
ζ	force constant for m-m collisions
ζ^*	force constant for m-m* collisions
ζ^{**}	force constant for m*-m* collisions
$\eta(c_x)$	function defined by Equation 12 of Chapter II
$\eta^*(c_x^*)$	function defined by Equation 24 of Chapter IV
$\eta(x,c_x,t)$	function defined by Equation 45 of Chapter II
$\bar{\eta}$	$\bar{\eta} = (\eta_1, \eta_2, \ldots, \eta_{2n})$
Θ	quantity proportional to mean free path, $\Theta = \frac{2}{\sqrt{\pi}} \ell$
Θ'	quantity proportional to mean free path, $\Theta' = \frac{\lambda + 2}{2} \Theta$
λ	free parameter in CT model equation
Λ	wavelength
Λ_o	equivalent wavelength
μ	viscosity coefficient for molecules of mass m

LIST OF SYMBOLS (Continued)

μ^*	viscosity coefficient for molecules for mass m*
ν	collision frequency
$\hat{\nu}$	nondimensional collision frequency, $\hat{\nu} = \nu/\omega_p$
ξ	transformed space coordinate
σ	complex frequency, $\sigma = \sigma_R + i\sigma_I$
$\hat{\sigma}$, $\hat{\sigma}_M$	nondimensional frequencies, $\hat{\sigma} = \sigma/\omega_o$, $\hat{\sigma}_M = \sigma/\omega_{Mo}$
$\hat{\sigma}_p$	nondimensional frequency, $\hat{\sigma}_p = \sigma/\omega_p$
τ	nondimensional time, $\tau = \omega t$
$\varphi(\bar{c})$	velocity dependent perturbation for molecules of mass m
$\varphi^*(\bar{c}^*)$	velocity dependent perturbation for molecules of mass m^*
$\Phi(x,\bar{c},t)$	time, space and velocity dependent perturbation for molecules of mass m
$\Phi^*(x,\bar{c}^*,t)$	time, space and velocity dependent perturbation for molecules of mass m*
χ,χ^*,χ^{**}	collision parameters
$\psi(c_x)$	function defined by Equation (11) of Chapter II
$\psi^*(c_x^*)$	function defined by Equation (23) of Chapter IV
$\psi(x,c_x,t)$	function defined by Equation 44 of Chapter II
ω	angular frequency
ω_o	characteristic frequency, $\omega_o = Ka_o$
ω_{Mo}	characteristic frequency, $\omega_{Mo} = KA_o$
ω_p	plasma frequency

SUMMARY

A new method for solving the problem of wave propagation in monatomic gases, gas mixtures and plasmas is presented. The method is the discrete ordinate method, which consists of approximating the integrations of the perturbation distribution function over velocity space by an appropriate quadrature. The accuracy of the method is established by solving the problems of forced and free sound wave propagation in a monatomic gas. The method is then applied to the very difficult problem of wave propagation in a binary gas mixture. Finally, it is demonstrated that the method may be applied to the study of plasma oscillations. In all cases, models of the Boltzmann collision integral were used. For the one-component systems, the collision integral was approximated by the models proposed by Bhatnagar, Gross and Krook [11] and by Cercignani and Tironi [33]. The model equation of Hamel [34] was used for the binary mixtures.

The results of this investigation may be summarized in the following conclusions:

1. If a normal mode assumption regarding the space and time dependence of the perturbation distribution function is made, the discrete ordinate technique yields results which are very similar to the polynomial expansion method. This was demonstrated for both free and forced sound wave propagation. For forced sound wave propagation in a simple monatomic gas, it was found that the n=8 discrete ordinate solution was comparable to the twenty-six term solution of Pekeris, et al. [6]. The n=8 solution was also comparable to a Grad 25 moment approximation [7]. The free

sound wave solution was similar to the Grad 13 moment theory [16]. The n=8 normal mode solution was in good agreement with the transformation theories for $r > 1$ (the collision frequency greater than the sound frequency). Excellent agreement with the experimental data of Meyer and Sessler [15] was obtained for $r > 1$.

2. A solution valid in a highly rarefied gas ($r \ll 1$) may be obtained by solving for the perturbation distribution function explicitly. The solution is in good agreement with experimental data [15] and other theories [7, 9, 10, 12] for $r < 0.1$. This method of solution also yields the transient development of the perturbation pressure field. It was found that the approach to a steady state is quite rapid.

3. A solution to the problem of forced sound propagation in a binary gas mixture was obtained which has the correct "classical" limits for $r \gg 1$ and has the correct dependence on the concentrations of the two components. The solution was in excellent agreement with the experimental data of Holmes and Tempest [19] for both Argon-Helium and Neon-Helium mixtures. A solution was also obtained for an Argon-Neon mixture.

4. The problem of free sound wave propagation in a binary mixture was also solved. There is no known theory with which this solution may be compared, however, the solution appears reasonable.

5. It is possible to use the method of discrete ordinates in the study of plasma oscillations. This was demonstrated by solving the problem of longitudinal electrostatic oscillations in a one-component plasma. The solution obtained was in good agreement with the theory of Bhatnagar, Gross and Krook [11]. It was found that the results predicted by the BGK and CT models were in very close agreement.

6. A major advantage of the discrete ordinate method is the ease with which higher approximations may be obtained. It appears that the method has great flexibility and may be adapted to many practical problems.

CHAPTER I

INTRODUCTION

Review of Recent Literature

In recent years considerable interest has been generated in kinetic theory descriptions of wave propagation in gases and plasmas. Theories concerning wave propagation based on macroscopic governing equations are strictly valid only when some characteristic frequency of the wave is much less than the collision frequency of the gas or plasma. This restriction is analogous to the requirement that the Knudsen number based on some characteristic length be small for gasdynamics problems to be described by macroscopic governing equations. If the characteristic frequency of the wave is much larger than the collision frequency of the gas, then the macroscopic equations must be replaced by the Boltzmann equation of kinetic theory. A further discussion of this point may be found in the survey article by Shermann and Talbot [1].

In the kinetic theory study of linearized wave propagation, two solution techniques have been used most frequently - the polynomial expansion method and the Fourier-Laplace transformation method.

Characteristic of the polynomial expansion method is the series expansion of the perturbation to a zero-order distribution function in an appropriate set of orthogonal polynomials. If the zero-order distribution function is Maxwellian, for example, Sonine-Legendre polynomials are used. Since in general an infinite number of terms are required to represent the perturbation exactly, one obtains an infinite system of partial differential equations for the coefficients of the polynomials. By taking the normal mode approach of assuming an exponential space and time dependence for the coefficients, the system of partial differential equations is reduced to an infinite system of algebraic equations. To complete the solution, a truncation of the system is necessary. The usual assumption is that all coefficients higher than a specified degree are zero. This truncation is characteristic of the Grad moment method [2]. A major disadvantage of the orthogonal polynomial method is that each approximation is independent of previous approximations. Because of this, higher approximations are increasingly difficult to obtain.

When using the Fourier-Laplace transformation method, one operates on the governing equation for the perturbation with a Laplace transform in time and a Fourier transform in space. After the usual algebraic manipulations, the perturbation may be obtained by inverting the Fourier-Laplace transform. Aside from an asymptotic analysis, such an inversion is not usually feasible. Even though it may not be possible to obtain the perturbation explicitly, the relation between the wave number and frequency of the wave (the dispersion relation) may be determined. Because of the complexity of the Fourier-Laplace solution, the dispersion relation must frequently be programmed for solution.

Rather than formally introducing the Fourier-Laplace transform, one may assume that the perturbation has a space and time dependence characteristic of a plane wave. The analysis leading to the dispersion relation is very similar to that used in the transformation method, and it may be shown that the dispersion relation is identical to the one obtained from the

Fourier-Laplace transformation (see, e.g., Reference [3]).

In the paragraphs which follow, a review of the recent literature on wave propagation will be given. In particular, the work (theoretical and experimental) on wave propagation in neutral gases, gas mixtures, and plasmas will be considered.

Waves in neutral gases are generally classified as either free or forced, depending on how they are created. Free wave refers to the wave which propagates as the result of an initial disturbance, while forced wave denotes a wave created by an external agent, for example, an oscillating piston. The former is an initial value problem, the latter a boundary value problem.

The first significant attempt to solve the problem of forced oscillations in a neutral gas using kinetic theory was made by Wang Chang and Uhlenbeck [4], who expanded the perturbation in a series of eigenfunctions of the linearized collision integral for Maxwellian molecules. Several approximations to the resultant infinite system of equations for the coefficients were considered, each of which failed to converge in the highly rarefied regime because of the manner in which the infinite system was truncated. The work of Wang Chang and Uhlenbeck was later extended by Pekeris et al., [5,6], who obtained much higher approximations for both Maxwellian and hard sphere molecules.

Kahn and Mintzer [7] avoided the convergence difficulties of Wang Chang and Uhlenbeck's solution by truncating the system of equations for the coefficients according to the Grad moment method. Since their results converged slowly in the highly rarefied regime, they argued that an expansion around a Maxwellian equilibrium was incorrect. By expanding around a distribution function which satisfied the collisionless Boltzmann equation, a solution for the highly rarefied regime was obtained. However, Toba [8] has noted that their collisionless distribution function was incorrect, and both Sirovich and Thurber [9], and Buckner and Ferziger [10] have demonstrated that excellent agreement with the experimental data in the highly rarefied regime may be obtained from the perturbation around a Maxwellian equilibrium distribution.

The propagation of sound in an isothermal gas was considered in the article [11] in which the BGK model* was postulated. More recently, the same problem was treated by Ostrowsky and Kleitman [3], who also demonstrated the equivalence of the results obtained by using the normal mode assumption with those obtained using the Fourier-Laplace transformation. Mason [12] considered the isothermal BGK model modified with a source term which included the effect of the oscillating boundary. Using the transformation method, he obtained results which were in qualitative agreement with the experimental data in the highly rarafied regime.

A solution has also been given by Holway [13], who treated the problem using the ellipsoidal distribution model of the Boltzmann equation. His solution, like many of the others, is in good agreement with the experimental data in the continuum regime and deviates from the data as the gas becomes rarefied. Also of interest is his discussion of the polynomial expansion method when the perturbation is around a Maxwellian equilibrium

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*Because of the complexity of the Boltzmann collision integral, many authors have used models of the Boltzmann equation. These models retain many of the characteristics of the full collision integral, but are considerably simpler mathematically. Perhaps the most familiar one is the single relaxation time model proposed by Bhatnagar, Gross, and Krook [11]. (This will be called the BGK model.)

distribution. It is shown that the expansion converges, but that the convergence is extremely slow when the ratio of collision frequency to sound frequency is small ($r < 1$).

Sirovich and Thurber [9] have presented one of the better solutions. By using a Fourier-Laplace transformation on model equations which may be derived from the linearized Boltzmann equation, they have obtained very good agreement with the experimental data from the continuum to the highly rarefied regime. It should be noted, however, that the highly rarafied solutions are obtained by effecting an analytic continuation of the dispersion relation in the less rarefied regime.

The best solution to date is that of Buckner and Ferziger [10]. Their analysis is based upon models of the Boltzmann collision integral. In the continuum regime the solution is obtained by assuming an appropriate space and time dependence for the perturbation. As mentioned earlier, this technique is similar to the transformation method. The results in the rarefied regime were obtained from the solution for the perturbation. Once the perturbation is known the pressure field may be calculated and, in turn, the pressure field yields the absorption and dispersion of the wave. The absorption and dispersion are in excellent agreement with the experimental data from the continuum to the highly rarefied regime.

The theoretical solutions may be compared with two sets of experimental data. The speed and absorption of sound were measured in various monatomic gases by Greenspan [14] and by Meyer and Sessler [15]. Both investigators held the sound frequency constant and varied the gas pressure. Greenspan used a quartz crystal oscillator and receiver at a frequency of 11,000 kc/sec, and Meyer and Sessler employed an electrostatic condenser transmitter and receiver at frequencies of 100 kc/sec and 200 kc/sec. The speed of the wave is obtained from the phase difference between a direct signal from the driving oscillator and the signal received at the receiver as a function of the sound path. The absorption of the sound disturbance was obtained from the logarithmic decrement in the sound level of the signal as a function of the sound path by varying the distance between the transmitter and receiver.

The propagation of an initial disturbance is one of the most fundamental problems in the kinetic theory of gases. However, there have been few attempts to solve this problem. Sirovich and Thurber [16] have investigated this initial value problem, again using kinetic models of the Boltzmann collision integral. In addition to the Fourier-Laplace solution, solutions obtained using the Grad moment method, the Navier-Stokes equations and the Burnett equations are presented. Buckner and Ferziger [17] have also treated this problem, but they present no numerical results. No experimental data are available with which to compare these solutions as yet.

Very little kinetic theory work has been done on wave propagation in gas mixtures, primarily because of the complexity of the governing equations. Liboff [18] treated the problem of forced wave propagation using a kinetic model which contained, in addition to the familiar relaxation term of the BGK model, coupling terms which were proportional to the temperature and velocity differences of the gases. However, rather than the kinetic equations, macroscopic equations developed by taking appropriate moments of the kinetic equations were used. Because these equations contained a large number of independent parameters, only special limiting cases were considered. As the parameters representing collision frequencies of the gases were not related to properties of the gases, no comparison with experiment was possible.

Theoretical solutions for gas mixtures may be compared with two sets of data. Holmes and Tempest [19] have measured the absorption and speed of sound in an A-He mixture, and the absorption in the mixtures Ne-He and Kr-He over the entire concentration range and at various pressures. Petralia [20] made similar measurements in an A-He mixture, but his results fail to agree with either those of Holmes and Tempest or macroscopic theory. Greenspan [21] has attributed this discrepancy to experimental error. It should be noted that experiments have been carried out in the nondispersive (continuum) region only.

No theoretical solutions or experimental data for the problem of free sound waves in gas mixtures could be found in the literature.

Wave propagation in a plasma is far more complex than in a neutral gas. Although a plasma is capable of a wide variety of oscillatory motions, the discussion here is restricted to longitudinal electrostatic modes. Other possible modes are discussed in Reference [22]. The discussion is further restricted to one-component plasmas.

The early microscopic studies of electrostatic oscillations in a one-component plasma by Vlasov [23] and by Landau [24] were based on the collisionless Boltzmann (or Vlasov) equation. It should be noted that many of the current investigations are also based on this equation.

The effects of collisions in a one-component plasma were considered by Bhatnagar, Gross, and Krook [11], who replaced the Boltzmann collision integral with a single relaxation time collision term. Using a Fourier-

Laplace transformation, they considered the evolution of an initial disturbance in an unbounded plasma. It was found that if the wave length of the disturbance were large compared to either the Debye length or the mean free path, a small change in frequency resulted as the collision frequency varied from zero to infinity. The accompanying absorption was small, reaching its maximum value when the collision frequency equaled the plasma frequency. For disturbances with wave lengths smaller than both the Debye length and the mean free path, a very high absorption occurred.

Discussion of the Method

The method of discrete ordinates was first used in the field of radiative transfer by Chandrasekhar [25]. Krook [26] suggested the application of this method of solution to the Boltzmann equation in 1955 and showed that for linear problems the method was closely related to the polynomial expansion method. However, it was not until recently that the method was applied to a wide class of gasdynamics problems. Huang and Giddens have used this solution technique to solve the linearized Couette flow problem (steady and unsteady) [27,28], the linearized channel flow problem [29] and the linearized Rayleigh problem [30,31]. It was found that the discrete method with an appropriate quadrature yields more accurate results over a wider range of Knudsen numbers for a given amount of computational effort than any of the other existing approximate analytical techniques. The method has recently been generalized by Huang [32] to apply to nonlinear problems.

For linear problems, the technique consists of replacing integrations over velocity space of the perturbation distribution function in the Boltzmann equation by an appropriate quadrature. The velocity dependence of the perturbation to the distribution function is thus approximated by a set of functions, each evaluated at appropriate discrete points in velocity space. Thus, rather than solving a single integro-differential equation for a function of space, time, and velocity, one solves a linear system of first order partial differential equations for a set of functions which are continuous in space and time, but are point-functions in velocity space. The solution to this system of partial differential equations is thus an approximation in the sense of a numerical truncation to the true distribution function.

Purpose of the Research

In this thesis the problem of wave propagation in monatomic gases, gas mixtures and plasmas will be investigated using the method of discrete ordinates. The investigation will be conducted in three phases.

First, the problem of forced and free sound propagation in a monatomic gas will be solved to determine the accuracy and limitations of the discrete ordinate method. These problems were chosen because many theories and good experimental data exist with which to compare the discrete ordinate solution. This analysis will be based upon the model equations proposed by Bhatnagar, Gross, and Krook (or the BGK model) [11] and by Cercignani and Tironi (or CT model) [33].

After the accuracy of the discrete ordinate method is established, the problem of forced and free sound propagation in gas mixtures will be treated using the model equation proposed by Hamel [34].

Finally, to demonstrate that the method of discrete ordinates may be applied to the study of plasma oscillations, longitudinal electrostatic waves in a one-component plasma will be investigated.

CHAPTER II

FORCED SOUND WAVE PROPAGATION IN A RAREFIED GAS

In this chapter the problem of forced oscillations in a simple monatomic gas will be considered using the model equations of Bhatnagar, Gross and Krook (the BGK model) [11] and of Cercignani and Tironi (the CT model) [33]. A normal mode analysis of the BGK and CT models will be performed. In addition, a solution valid in a highly rarefied gas will be obtained by solving the BGK model for the perturbation distribution function explicitly. A discussion of the limitations of the normal mode solution will be given later.

The physical problem is shown in Figure 1. A semi-infinite, monatomic gas of mass m with an equilibrium temperature T_o, pressure p_o and number density n_o is disturbed by a harmonically oscillating piston. The piston oscillates normal to the y-z plane with an angular frequency ω. The amplitude and velocity of the oscillation are assumed small.

Theoretical Formulation (Normal Mode Solution)

BGK Model

Discussion of the Model Equation. In this section the governing equation is the linearized, single relaxation time collision model postulated by Bhatnagar, Gross and Krook [11]. This is perhaps the simplest and most useful model of the Boltzmann equation and may be shown to be a first approximation to the linearized Boltzmann collision integral for Maxwellian molecules (see, e.g., Reference [9]).

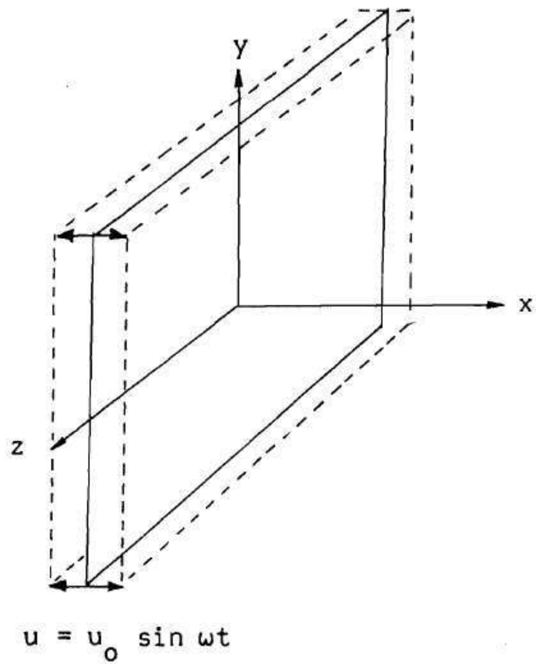

Figure 1. Geometry and Coordinate System for Forced Sound Propagation.

In the absence of body forces the equation is [11]

$$\beta_o \frac{\partial\Phi}{\partial t} + c_x \frac{\partial\Phi}{\partial x} = \frac{1}{\theta}\left[-\Phi + n' + 2c_x q_x' + (c^2 - \tfrac{3}{2})T'\right] , \tag{1}$$

where Φ is the perturbation of the distribution function, f, from an absolute Maxwellian distribution and is defined by

$$f = \beta_o^3 f_o[1 + \Phi(x,\bar{c},t)], \tag{2}$$

where

$$\beta_o = \left(\frac{m}{2kT_o}\right)^{1/2} \tag{3}$$

and

$$f_o = n_o \pi^{-3/2} e^{-c^2} . \tag{4}$$

The molecular mass is denoted by m; k is Boltzmann's constant and $\bar{c}$ is the molecular velocity nondimensionalized by the most probable molecular speed, $1/\beta_o$. The quantities n', q_x' and T' are the perturbation number density, macroscopic velocity and temperature, respectively, and are given by

$$n' = \pi^{-3/2} \int e^{-c_1^2} \Phi(x,\bar{c}_1,t) d^3c_1 \tag{5}$$

$$q_x' = \pi^{-3/2} \int e^{-c_1^2} c_{x_1} \Phi(x,\bar{c}_1,t) d^3c_1 \tag{6}$$

and

$$T' = \frac{2}{3}\pi^{-3/2}\int e^{-c_1^2}(c_1^2 - \frac{3}{2})\Phi(x,\bar{c}_1,t)d^3c_1 \tag{7}$$

where n', q_x' and T' are normalized by n^o, $1/\beta_o$ and T_o, respectively, and the shorthand notation $\int d^3c_1 = \int_{-\infty}^{+\infty} dc_{x_1} \int_{-\infty}^{+\infty} dc_{y_1} \int_{-\infty}^{+\infty} dc_{z_1}$ is used. The parameter Θ is related to the mean free path, ℓ, by

$$\Theta = \frac{2}{\sqrt{\pi}}\ell \quad . \tag{8}$$

Normal Mode Assumption. Since in this section a normal mode solution is being sought, it is assumed that

$$\Phi(x,\bar{c},t) = \varphi(\bar{c})e^{\hat{i}(\omega t - Kx)} \quad , \tag{9}$$

where φ is a function of molecular velocity alone, $\hat{i} = \sqrt{-1}$, ω is the angular frequency and K is the complex wave number. The problem is to determine K for a given frequency ω. After K is found, the phase velocity of the wave is given by $\frac{\omega}{K_R}$, and the absorption by K_I, where K_R and K_I are the real and imaginary parts of K, respectively. Physically, $\frac{1}{K_I}$ is the distance required for the wave to attenuate to 1/e of its initial amplitude.

By substituting the assumed solution into Equation (1), introducing Equations (5), (6) and (7), cancelling the exponential term and rearranging, one obtains the integral equation

$$\begin{aligned}(\hat{i}\omega\Theta\beta_o - \hat{i}c_x\Theta K + 1)\varphi(\bar{c}) &= \pi^{-3/2}\int e^{-c_1^2}\varphi(\bar{c}_1)d^3c_1 \\ &+ 2c_x\pi^{-3/2}\int e^{-c_1^2} c_{x_1}\varphi(\bar{c}_1)d^3c_1\end{aligned} \tag{10}$$

$$+ \frac{2}{3} (c^2 - \frac{3}{2})\pi^{-3/2} \int e^{-c_1^2} (c_1^2 - \frac{3}{2})\, \varphi(\bar{c}_1) d^3 c_1 \quad .$$

<u>Transformation of Integral Equation</u>. The integrals in Equation (10) are three-fold, the range of each integration being from $-\infty$ to $+\infty$. If each integration is approximated by a quadrature using n positive discrete points and an equal number of negative points, there results a system of $8n^3$ equations for the perturbation evaluated at $8n^3$ different points in velocity space. Fortunately, it is not necessary to deal with systems of this magnitude. Two of the three integrations may be eliminated by introducing the functions

$$\psi(c_x)\ \pi^{-3/2} \int_{-\infty}^{+\infty} dc_y \int_{-\infty}^{+\infty} dc_z e^{-c_y^2 - c_z^2} \ \varphi(\bar{c}) \tag{11}$$

and

$$\eta(c_x) = \pi^{-3/2} \int_{-\infty}^{+\infty} dc_y \int_{-\infty}^{+\infty} dc_z e^{-c_y^2 - c_z^2} (c_y^2 + c_z^2)\varphi(\bar{c}) \quad . \tag{12}$$

This type of transformation was first used by van Kampen [35] in the study of oscillations in a one-component, collisionless plasma. Later, Bernstein, Trehan and Weenick [36] treated the same problem and demonstrated that the results obtained from the transformed system are identical to those obtained from the original equation. Because collisions were neglected in both of these studies, it was only necessary to introduce the first function. More recently, both Bienkowski [37] and Chu [38] have used this transformation in the analysis of a number of gas dynamics problems.

In terms of $\psi(c_x)$ and $\eta(c_x)$, Equation (10) becomes

$$(\hat{i}\omega\Theta\beta_o - \hat{i}c_x\Theta K + 1)\varphi(\bar{c}) = \int_{-\infty}^{+\infty} e^{-c_{x_1}^2} \psi(c_{x_1})dc_{x_1} \tag{13}$$

$$+ 2c_x \int_{-\infty}^{+\infty} e^{-c_{x_1}^2} c_{x_1}\psi(c_{x_1})dc_{x_1} + \frac{2}{3}(c^2 - \frac{3}{2}) \cdot$$

$$\cdot \int_{-\infty}^{+\infty} e^{-c_{x_1}^2} [\eta(c_{x_1}) + (c_{x_1}^2 - \frac{3}{2})\psi(c_{x_1})]dc_{x_1}$$

The transformation is completed by integrating Equation (13) over (c_y, c_z) using the two weighting functions $e^{-c_y^2-c_z^2}$ and $(c_y^2 + c_z^2)e^{-c_y^2-c_z^2}$. Multiplying Equation (13) by $e^{-c_y^2-c_z^2}$ and performing the (c_y, c_z) integration yield

$$\sqrt{\pi}\ (\hat{i}\omega\Theta\beta_o - \hat{i}c_x\Theta K + 1)\psi(c_x) = \int_{-\infty}^{+\infty} e^{-c_{x_1}^2} \psi(c_{x_1})dc_{x_1} \tag{14}$$

$$+ 2c_x \int_{-\infty}^{+\infty} e^{-c_{x_1}^2} c_{x_1}\psi(c_{x_1})dc_{x_1}$$

$$+ \frac{2}{3}(c_x^2 - \frac{1}{2})\int_{-\infty}^{+\infty} e^{-c_{x_1}^2} [\eta(c_{x_1}) + (c_{x_1}^2 - \frac{3}{2})$$

$$\cdot \psi(c_{x_1})]dc_{x_1}$$

Multiplying Equation (13) by $(c_y^2 + c_z^2)e^{-c_y^2-c_z^2}$ and performing the same integration yield

$$\sqrt{\pi}\,(\hat{i}\omega\Theta\beta_o - \hat{i}c_x\Theta K + 1)\eta(c_x) \tag{15}$$

$$= \int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\,\psi(c_{x_1})dc_{x_1} + 2c_x \int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\,c_{x_1}\psi(c_{x_1})dc_{x_1}$$

$$+ \frac{2}{3}\,(c_x^2 + \frac{1}{2}) \int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\,[\eta(c_{x_1}) + (c_{x_1}^2 - \frac{3}{2})\psi(c_{x_1})]dc_{x_1}$$

The use of $\psi(c_x)$ and $\eta(c_x)$ thus transforms the single integral equation involving a three-fold integration into a coupled pair of integral equations involving a single integration. Accordingly, when the integrals are approximated by an appropriate quadrature, the order of the resultant system is $4n$ rather than $8n^3$.

Introduction of Forced Sound Wave Parameter. Now consider the term

$$(\hat{i}\omega\Theta\beta_o - \hat{i}c_x\Theta K + 1)$$

in more detail. The parameter β_o is related the adiabatic speed of sound,

$$a_o = (\gamma\,\frac{kT_o}{m})^{1/2} \quad ,$$

by

$$\beta_o = \sqrt{\frac{\gamma}{2}}\;\frac{1}{a_o} \quad , \tag{16}$$

where γ is the ratio of specific heats (equal to 5/3 for a monatomic gas). For the purpose of defining a Knudsen number, a wavelength, Λ_o, is introduced:

$$\Lambda_o = \frac{2\pi a_o}{\omega} \quad . \tag{17}$$

Using Equations (8), (16) and (17), one may write

$$(\hat{i}\omega\Theta\beta_o - \hat{i}c_x\Theta K + 1) = \hat{i}4\left(\frac{5\pi}{6}\right)^{1/2} K_n - \hat{i}4\sqrt{\pi}\, c_x \frac{Ka_o}{\omega} K_n + 1 \tag{18}$$

where

$$K_n = \frac{\ell}{\Lambda_o} \quad . \tag{19}$$

For the problem of forced sound wave propagation, it is customary to use the parameter

$$r = \frac{p_o}{\mu\omega} \quad , \tag{20}$$

where p_o is the equilibrium pressure, μ the viscosity coefficient and ω the angular frequency, rather than the Knudsen number, K_n, based on the wavelength Λ_o. The two parameters are related through the following equation

$$K_n = \frac{8}{5\pi(2\pi\gamma)^{1/2}} \frac{1}{r} \quad . \tag{21}$$

As noted in Reference [9], Equation (21) is obtained by using the mean free path for hard sphere molecules. Physically, r is a measure of the ratio of the collision frequency to the sound frequency. Using equation (21), one obtains

$$(\hat{i}\omega\Theta\beta_o - \hat{i}c_x\Theta K + 1) = \hat{i}\,\frac{16}{5\pi}\,\frac{1}{r} - \hat{i}\sqrt{6/5}\;\frac{16}{5\pi}\,\frac{1}{r}\,c_x\hat{K} + 1 \quad , \tag{22}$$

where $\hat{K} = \frac{Ka_o}{\omega}$ is a nondimensional wave number. Note that

$$\hat{K}_R = \frac{a_o}{a}$$

and

$$\hat{K}_I = \frac{K_I a_o}{\omega} \; .$$

Substitution of Equation (22) into Equations (14) and (15) yields

$$\sqrt{\pi}\,(\hat{i}\,\frac{16}{5\pi}\,\frac{1}{r} - \hat{i}\sqrt{6/5}\,\frac{16}{5\pi}\,\frac{1}{r}\,c_x\hat{K} + 1)\psi(c_x) = \tag{23}$$

$$\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\,\psi(c_{x_1})dc_{x_1} + 2c_x\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\,c_{x_1}\psi(c_{x_1})dc_{x_1}$$

$$+\,\frac{2}{3}\,(c_x^2 - \frac{1}{2})\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\,[\eta(c_{x_1}) + (c_{x_1}^2 - \frac{3}{2})\psi(c_{x_1})]dc_{x_1}$$

and

$$\sqrt{\pi}\,(\hat{i}\,\frac{16}{5\pi}\,\frac{1}{r} - \hat{i}\sqrt{6/5}\,\frac{16}{5\pi}\,\frac{1}{r}\,c_x\hat{K} + 1)\eta(c_x) = \tag{24}$$

$$\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\,\psi(c_{x_1})dc_{x_1} + 2c_x\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\,c_{x_1}\psi(c_{x_1})dc_{x_1}$$

$$+\,\frac{2}{3}\,(c_x^2 + \frac{1}{2})\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\,[\eta(c_{x_1}) + (c_{x_1}^2 - \frac{3}{2})\psi(c_{x_1})]dc_{x_1}$$

Discrete Ordinate Solution. The simultaneous solution of Equations (23) and (24) by the discrete ordinate method requires the approximation of the various integrals by finite summations. That is, one writes, for example,

$$\int_{-\infty}^{+\infty} e^{-c_{x_1}^2} \eta(c_{x_1})dc_{x_1} \cong \sum_{i=1}^{2n} H_i \eta(c_i) \quad ,$$

where H_i is the i^{th} weighting coefficient and c_i is the i^{th} discrete velocity point at which the function η is evaluated. In general, H_i and c_i depend upon the order of the approximation desired, the weighting function of the integral and the limits of the integration.

It has been well established that the modified Gaussian quadrature yields the most accurate results in the rarefied regime (see, e.g., References [27] and [29]). For this reason, the modified quadrature will be used in all subsequent calculations. The c_i and H_i for the modified quadrature are listed in Table 1 for values of n from 1 to 8. These values were obtained from Reference [39].

Applying the discrete ordinate method to Equations (23) and (24) yields

$$\left(\sqrt{5/6}\,\frac{1}{c_\ell} - \hat{i}\,\frac{5\pi}{16}\sqrt{5/6}\,\frac{r}{c_\ell}\right)\psi_\ell + \hat{i}\,\frac{5}{16}\sqrt{5\pi/6}\,\frac{r}{c_\ell}\sum_{i=1}^{2n} H_i\Big[1 + 2c_\ell c_i + \tfrac{2}{3}(c_\ell^2 - \tfrac{1}{2})(c_i^2 - \tfrac{3}{2})\Big]\psi_i + \hat{i}\,\frac{5}{16}\sqrt{5\pi/6}\,\frac{r}{c_\ell}\sum_{i=1}^{2n} H_i\Big[\tfrac{2}{3}(c_\ell^2 - \tfrac{1}{2})\Big]\eta_i = \hat{K}\psi_\ell \tag{25}$$

Table 1. Discrete Velocities and Weighting Coefficients

n	c_i	H_i
1	0.564189584	0.886226925
2	0.300193931	0.640529180
	1.252421045	0.245697746
3	0.190554150	0.446029770
	0.848251867	0.396468267
	1.799776578	0.437288880
4	0.133776447	0.325302999
	0.624324689	0.421107102
	1.342537825	0.133442501
	2.262664477	$0.637432347 \times 10^{-2}$
5	0.100242151	0.248406139
	0.482813937	0.392331068
	1.060949843	0.211418215
	1.779729480	$0.332466497 \times 10^{-1}$
	2.669760263	$0.824853772 \times 10^{-3}$
6	$0.786006456 \times 10^{-1}$	0.196849206
	0.386739236	0.349155692
	0.866431154	0.257257962
	1.465691545	$0.760135170 \times 10^{-1}$
	2.172716470	$0.685210492 \times 10^{-2}$
	3.036815972	$0.984442006 \times 10^{-4}$
7	$0.637163079 \times 10^{-1}$	0.160614533
	0.318193583	0.306303720
	0.724183347	0.275545914
	1.238085494	0.120624443
	1.838440401	$0.218895217 \times 10^{-1}$
	2.531553434	$0.123780543 \times 10^{-2}$
	3.373437976	$0.109889525 \times 10^{-4}$
8	$0.529778530 \times 10^{-1}$	0.134163250
	0.267409313	0.268130722
	0.616160988	0.276228985
	1.064704717	0.157293957
	1.587766508	$0.448128001 \times 10^{-1}$
	2.185429830	$0.540120585 \times 10^{-2}$
	2.862054102	$0.194398570 \times 10^{-3}$
	3.686276823	$0.160745515 \times 10^{-5}$

and

$$\hat{i}\,\frac{5\pi}{16}\,\sqrt{5/6}\,\frac{r}{c_\ell}\sum_{i=1}^{2n} H_i[1 + 2c_\ell c_i + \frac{2}{3}(c_\ell^2 + \frac{1}{2})(c_i^2 - \frac{3}{2})]\psi_i \tag{26}$$

$$+ \hat{i}\,\frac{5}{16}\sqrt{5\pi/6}\,\frac{r}{c_\ell}\sum_{i=1}^{2n} H_i[\frac{2}{3}(c_\ell^2 + \frac{1}{2})]\eta_i$$

$$+ (\sqrt{5/6}\,\frac{1}{c_\ell} - \hat{i}\,\frac{5\pi}{16}\,\sqrt{5/6}\,\frac{r}{c_\ell})\eta_\ell = \hat{K}\eta_\ell$$

where

n = number of positive discrete velocity points,

H_i = i^{th} weighting function,

c_i = i^{th} discrete velocity point on c_{x_1} axis,

c_ℓ = ℓ^{th} discrete velocity point on c_x axis,

$\psi_i = \psi(c_i)$,

$\psi_\ell = \psi(c_\ell)$,

$\eta_i = \eta(c_i)$,

and

$\eta_\ell = \eta(c_\ell)$.

The following convention has been assumed for the indexes:

$c_i < 0$, $i = 1, 2, \ldots, n$,

$c_i > 0$, $i = n+1, n+2, \ldots, 2n$,

$$c_i = c_{2n+1-i}$$

$$c_i = c_\ell \quad \text{if} \quad i = \ell \,,$$

$$H_i > 0 \quad , \qquad\qquad i = 1, 2, \ldots, 2n$$

and

$$H_i = H_{2n+1-i}$$

By letting $\ell = 1, 2, \ldots, 2n$, one obtains a linear homogeneous system of algebraic equations of order 4n in the 4n unknowns (ψ_1, ψ_2, ..., ψ_{2n}, η_1, η_2, ..., η_{2n}). This system of equations is of the form

$$\bar{\bar{A}}\begin{pmatrix}\bar{\psi}\\ \bar{\eta}\end{pmatrix} = \hat{K}\bar{\bar{I}}\begin{pmatrix}\bar{\psi}\\ \bar{\eta}\end{pmatrix} \,, \tag{27}$$

where

$$\begin{pmatrix}\bar{\psi}\\ \bar{\eta}\end{pmatrix} = \begin{vmatrix}\psi_1\\ \psi_2\\ \cdot\\ \cdot\\ \cdot\\ \psi_{2n}\\ \eta_1\\ \eta_2\\ \cdot\\ \cdot\\ \cdot\\ \eta_{2n}\end{vmatrix} \tag{28}$$

$\bar{\bar{I}}$ = Identity matrix

and

$\bar{\bar{A}}$ is the matrix with elements

$$A_{\ell i} = \left(\sqrt{5/6}\,\frac{1}{c_\ell} - \hat{i}\,\frac{5\pi}{16}\sqrt{5/6}\,\frac{r}{c_\ell}\right)\delta_{\ell i} + \hat{i}\,\frac{5}{16}\sqrt{5\pi/6}\,\frac{r}{c_\ell}\,H_i \tag{29}$$

$$\cdot\left[1 + 2c_\ell c_i + \frac{2}{3}\left(c_\ell^2 - \frac{1}{2}\right)\left(c_i^2 - \frac{3}{2}\right)\right],$$

$$A_{\ell,i+2n} = \hat{i}\left[\frac{5}{16}\sqrt{5\pi/6}\,\frac{r}{c_\ell}\,H_i\,\frac{2}{3}\left(c_\ell^2 - \frac{1}{2}\right)\right], \tag{30}$$

$$A_{\ell+2n,i} = \hat{i}\,\frac{5}{16}\sqrt{5\pi/6}\,\frac{r}{c_\ell}\,H_i\left[1 + 2c_\ell c_i + \frac{2}{3}\left(c_\ell^2 + \frac{1}{2}\right)\left(c_i^2 - \frac{3}{2}\right)\right], \tag{31}$$

$$A_{\ell+2n,i+2n} = \left(\sqrt{5/6}\,\frac{1}{c_\ell} - \hat{i}\,\frac{5\pi}{16}\sqrt{5/6}\,\frac{r}{c_\ell}\right)\delta_{\ell i} \tag{32}$$

$$+ \hat{i}\,\frac{5}{16}\sqrt{5\pi/6}\,\frac{r}{c_\ell}\,H_i\left[\frac{2}{3}\left(c_\ell^2 + \frac{1}{2}\right)\right],$$

where

$$\ell = (1, 2, \ldots, 2n),$$

$$i = (1, 2, \ldots, 2n).$$

A non-trivial solution will exist for this system if the determinant of the coefficient matrix, $(\bar{\bar{A}} - \hat{K}\bar{\bar{I}})$, is zero. Since this determinant is zero if $\hat{K}$ is an eigenvalue of $\bar{\bar{A}}$, the phase velocity and absorption of the sound wave can be found once the eigenvalues of $\bar{\bar{A}}$ are calculated.

<u>CT Model</u>

<u>Discussion of the Model Equation</u>. The analysis of this section is

based upon the model equation recently proposed by Cercignani and Tironi [33]. This model is derived by the method suggested by Gross and Jackson [41]. It has been applied to several linearized gas dynamics problems and shown to yield more accurate results than the BGK model. It should be noted that this equation can be made to predict the correct Prandtl number for a monatomic gas, in contrast to the BGK model which predicts a Prandtl number of 1.

In the absence of body forces, the CT model of the Boltzmann equation is written [33]

$$\beta_o \frac{\partial\Phi}{\partial t} + c_x \frac{\partial\Phi}{\partial x} = \frac{2}{(\lambda+2)\Theta}[-\Phi + n' + 2c_x q_x' + (c^2 - \tfrac{3}{2})T' - \lambda c_i c_j p_{ij}' + \tfrac{\lambda}{2} c^2(n' + T')] \tag{33}$$

The constant λ may be adjusted so as to predict the correct Prandtl number for the gas ($Pr = \frac{2}{\lambda+2}$). Note that Equation (33) reduces to the BGK model for $\lambda = 0$. The perturbation pressure term is denoted by p_{ij}'. For a plane wave propagating in the x-direction, one may show

$$c_i c_j p_{ij}' = c_x^2 \pi^{-3/2} \int e^{-c_1^2} c_{x_1}^2 \Phi(x, \bar{c}_1, t) d^3 c_1 + \frac{1}{2}(c_y^2 + c_z^2)\pi^{-3/2} \int e^{-c_1^2} (c_{y_1}^2 + c_{z_1}^2)\Phi(x, \bar{c}_1, t) d^3 c_1 \tag{34}$$

Normal Mode Formulation. The formulation from this point is identical to the formulation for the BGK model and is therefore sketched very briefly. The substitution of Equations (5), (6), (7) and (34) into

Equation (33) yields the governing equation for the perturbation $\Phi(x,\bar{c},t)$. By introducing the normal mode assumption, Equation (9), an equation for $\varphi_{(\bar{c})}$ is obtained. This equation may be transformed to the following pair of coupled equations

$$\sqrt{\pi}\left(\hat{i}\,\frac{8(\lambda+2)}{5\pi}\,\frac{1}{r} - \hat{i}\,\sqrt{6/5}\,\frac{8(\lambda+2)}{5\pi}\,c_x\hat{K}\,\frac{1}{r} + 1\right)\psi(c_x) = \tag{35}$$

$$\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\,\psi(c_{x_1})dc_{x_1} + 2c_x\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\,c_{x_1}\psi(c_{x_1})dc_{x_1}$$

$$+\frac{2}{3}\left(c_x^2 - \frac{1}{2}\right)\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\left[\eta(c_{x_1}) + \left(c_{x_1}^2 - \frac{3}{2}\right)\psi(c_{x_1})\right]dc_{x_1}$$

$$+\lambda\left[\left(-\frac{2}{3}\,c_x^2 + \frac{1}{3}\right)\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\,c_{x_1}^2\psi(c_{x_1})dc_{x_1}\right.$$

$$\left.+\left(\frac{1}{3}\,c_x^2 - \frac{1}{6}\right)\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\,\eta(c_{x_1})dc_{x_1}\right]$$

and

$$\sqrt{\pi}\left(\hat{i}\,\frac{8(\lambda+2)}{5\pi}\,\frac{1}{r} - \hat{i}\,\sqrt{6/5}\,\frac{8(\lambda+2)}{5\pi}\,c_x\hat{K}\,\frac{1}{r} + 1\right)\eta(c_x) = \tag{36}$$

$$\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\,\psi(c_{x_1})dc_{x_1} + 2c_x\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\,c_{x_1}\psi(c_{x_1})dc_{x_1}$$

$$+\frac{2}{3}\left(c_{x_1}^2 + \frac{1}{2}\right)\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\left[\eta(c_{x_1}) + \left(c_{x_1}^2 - \frac{3}{2}\right)\psi(c_{x_1})\right]dc_{x_1}$$

$$+ \lambda\Big[(-\tfrac{2}{3}c_x^2+\tfrac{2}{3})\int_{-\infty}^{+\infty} e^{-c_{x_1}^2} c_{x_1}^2 \psi(c_{x_1})dc_{x_1}$$

$$+ (\tfrac{1}{3}c_x^2-\tfrac{1}{2})\int_{-\infty}^{+\infty} e^{-c_{x_1}^2} \eta(c_{x_1})dc_{x_1}\Big],$$

where $\psi(c_x)$ and $\eta(c_x)$ are given by Equations (11) and (12), respectively. By approximating the integrals in Equations (35) and (36) by finite summations, one obtains a system of equations of the form

$$\bar{\bar{A}}\binom{\bar{\psi}}{\bar{\eta}} = \bar{\bar{I}}\hat{K}\binom{\bar{\psi}}{\bar{\eta}} ,$$

where $\bar{\bar{A}}$ has the elements

$$A_{\ell i} = (\sqrt{5/6}\,\frac{1}{c_\ell} - \hat{\imath}\,\frac{5\pi\sqrt{5/6}}{8(\lambda+2)}\frac{r}{c_\ell})\delta_{\ell i} + \hat{\imath}\,\frac{5}{8(\lambda+2)}\sqrt{5\pi/6}\,\frac{r}{c_\ell}H_i\Big[1 \tag{37}$$

$$+ 2c_\ell c_i + \tfrac{2}{3}(c_\ell^2-\tfrac{1}{2})(c_i^2-\tfrac{3}{2})+\lambda(-\tfrac{2}{3}c_\ell^2+\tfrac{1}{3})c_i^2\Big] ,$$

$$A_{\ell,i+2n} = \hat{\imath}\,\frac{5}{8(\lambda+2)}\sqrt{5\pi/6}\,\frac{r}{c_\ell}H_i\Big[\tfrac{2}{3}(c_\ell^2-\tfrac{1}{2}) + \lambda(\tfrac{1}{3}c_\ell^2-\tfrac{1}{6})\Big], \tag{38}$$

$$A_{\ell+2n,i} = \hat{\imath}\,\frac{5}{8(\lambda+2)}\sqrt{5\pi/6}\,\frac{r}{c_\ell}H_i\Big[1 + 2c_\ell c_i + \tfrac{2}{3}(c_\ell^2+\tfrac{1}{2})(c_i^2-\tfrac{3}{2})$$

$$+ \lambda(-\tfrac{2}{3}c_\ell^2+\tfrac{2}{3})c_i^2\Big] \tag{39}$$

and

$$A_{\ell+2n,i+2n} = \left(\sqrt{5/6}\,\frac{1}{c_\ell} - \hat{i}\,\frac{5\pi\sqrt{5/6}}{8(\lambda + 2)}\frac{r}{c_\ell}\right)\delta_{\ell i} \qquad (40)$$

$$+ \hat{i}\,\frac{5}{8(\lambda + 2)}\,\sqrt{5\pi/6}\,\frac{r}{c_\ell}\,H_i\left[\frac{2}{3}(c_\ell^2 + \frac{1}{2}) + \lambda(\frac{1}{3}\,c_\ell^2 - \frac{1}{3})\right]$$

The absorption and dispersion are again obtained by calculating the eigenvalues of $\bar{\bar{A}}$.

Computational Procedure (Normal Mode Solution)

Once values of n and r (and λ for the CT model) have been specified, a simple variation of ℓ and i completely determines the matrix $\bar{\bar{A}}$. To find the eigenvalues of $\bar{\bar{A}}$, the matrix is first reduced to a tridiagonal matrix. The calculation* involves a number of pre- and post-multiplications of the matrix $\bar{\bar{A}}$ by certain matrices which depend upon the elements of $\bar{\bar{A}}$. A complete discussion of the procedure used may be found in Reference [42].

When the tridiagonal matrix has been obtained, a straight-forward calculation yields the characteristic polynomial. The eigenvalues of $\bar{\bar{A}}$ are determined once the zeroes of this polynomial are computed. To calculate the zeroes, a variation of Ward's downhill method was used. For the details of this method, see Reference [43].

Discussion of Results (Normal Mode Solution)

In this section the discrete ordinate normal mode solutions are compared with solutions obtained using other techniques and with the experimental data of Meyer and Sessler [15]. The experimental results of

*All calculations were performed on a Burrough's B5500 digital computer.

Greenspan [14] are in very close agreement with those of Reference [15] and are not presented.

Discrete ordinate solutions were obtained for $n = 4$ to 8. The parameter r, the ratio of collision frequency to sound frequency, was varied from 0.1 to 100. All calculations for the CT model are for $\lambda = 1.0$, the value which gives a Prandtl number of 2/3.

It was found that for $r > 2$, excellent convergence was obtained for the absorption and dispersion coefficients for both models. The range of convergence may be extended to smaller values of r by increasing n, however, the extension is not significant (the $n = 6$ solution diverges from the $n = 8$ solution at $r = 1.25$). This slow convergence is not surprising when one considers the similarity between the discrete ordinate solution and the polynomial expansion method (it has been shown by Holway [13] that the convergence of the polynomial expansion solution is extremely slow when the collision frequency of the gas is comparable to the frequency of the sound wave).

The $n = 8$ solutions for the absorption coefficient, $\frac{-K_I a_o}{\omega}$, and the dispersion coefficient, $\frac{a_o}{a}$, are compared with the experimental data of Meyer and Sessler [15] in Figures 2 and 3. Note that excellent agreement with the experimental data is obtained for $r > 1$ for both the BGK and CT models. The BGK model slightly underestimates the absorption, but this has been observed by other investigators [10] and may be attributed to the incorrect Prandtl number predicted by the BGK model. For $r < 1$, the comparison with experiment is not very good.

There are two possible explanations for this disagreement with the experimental data for $r < 1$. Both Sirovich and Thurber [9] and Buckner and

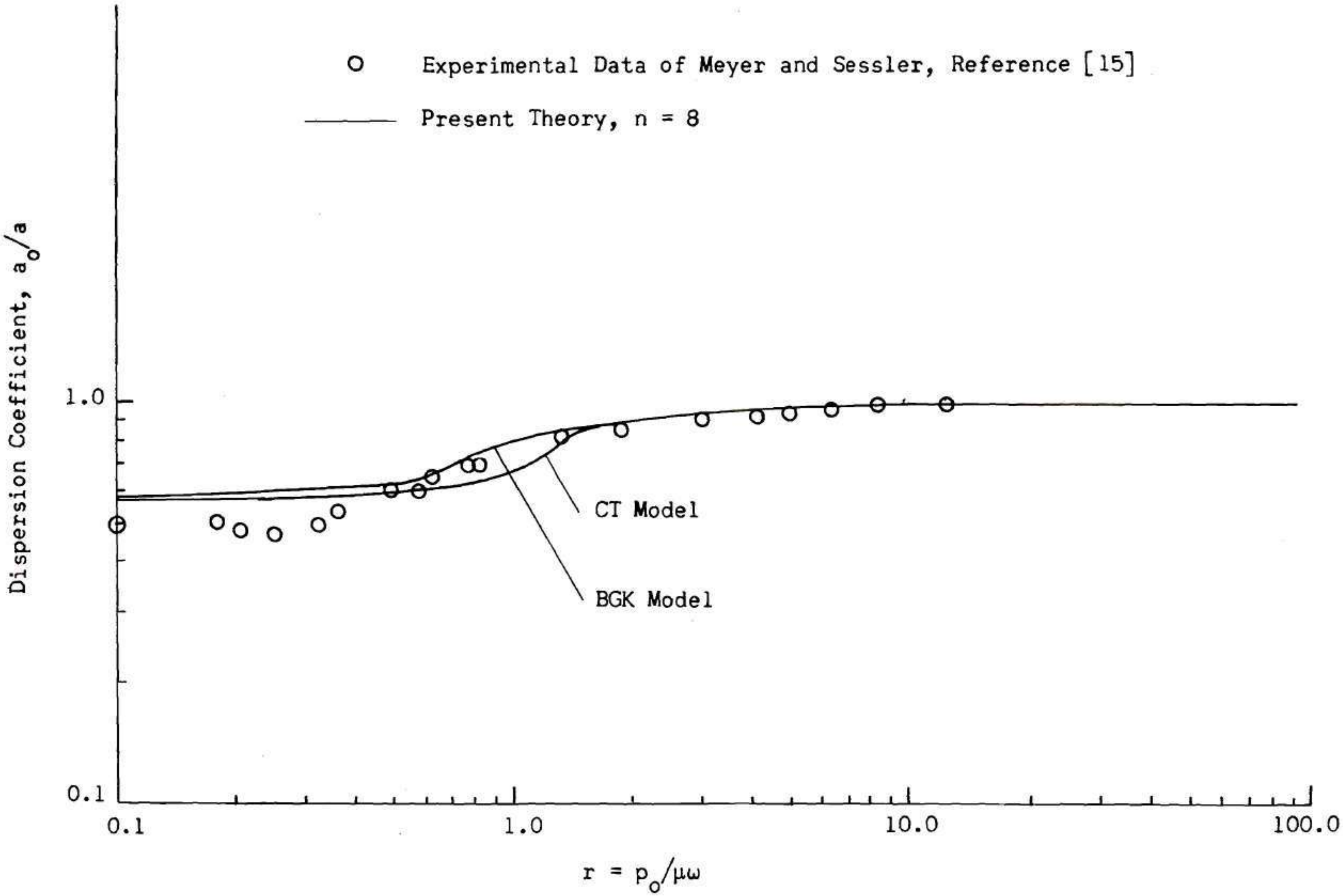

Figure 2. Comparison of Discrete Ordinate n = 8 Solution for the Dispersion Coefficient with Experimental Data.

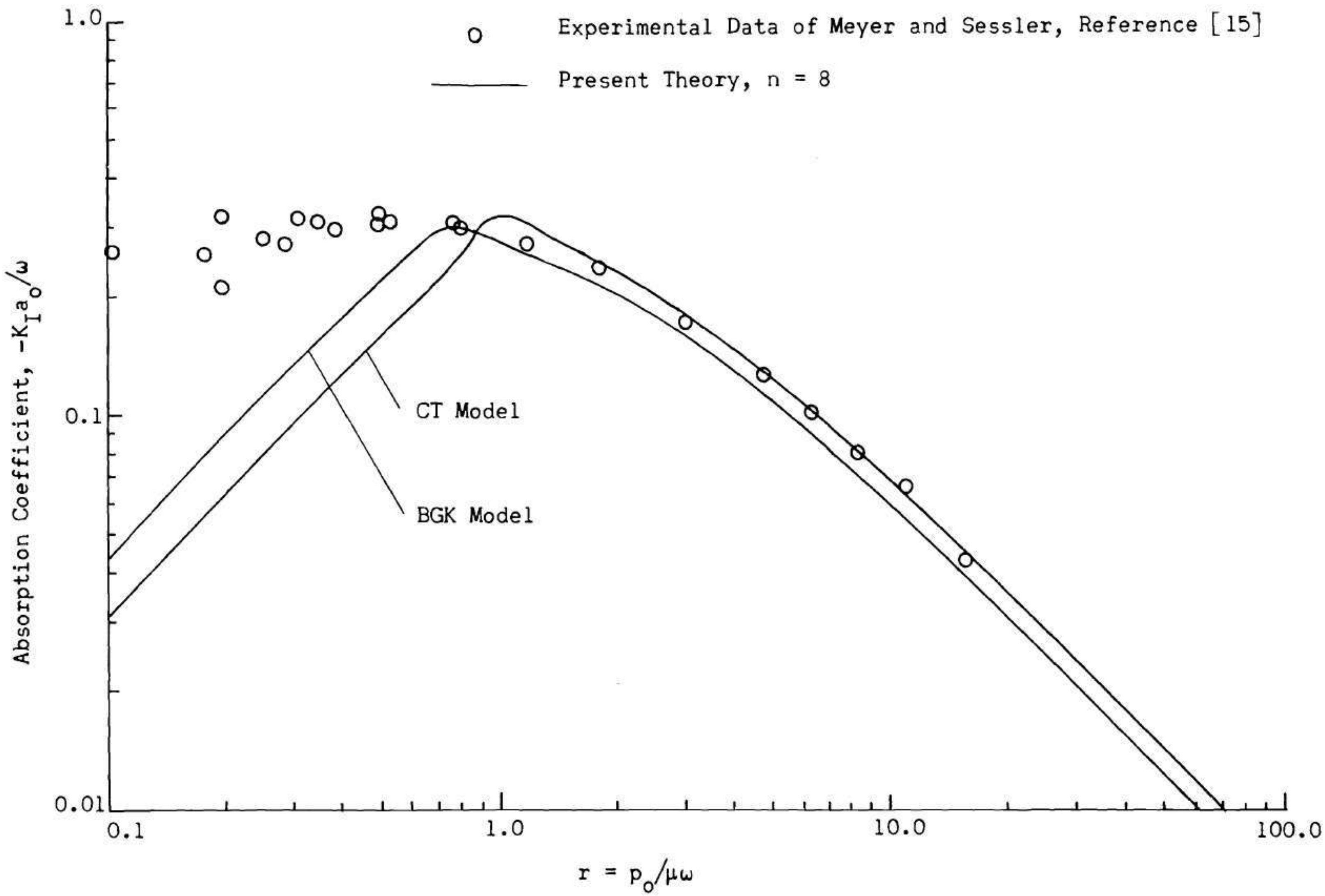

Figure 3. Comparison of Discrete Ordinate n = 8 Solution for the Absorption Coefficient With Experimental Data.

and Ferziger [10] have shown that for models of the Boltzmann equation there exist critical values of r beyond which no dispersion relation exists ($r \cong 0.74$ for the BGK model). These critical values of r depend upon the model equation, however, and there are no observable discontinuities in measurable properties at the critical value of r. A physical explanation has been given by Greenspan [21], who suggested that it is probable that modes other than the sound mode are observed when r is small. The discrete ordinate solution tends to substantiate this latter proposal. For $r < 1$, it was found that there were several modes with damping comparable to the sound mode. It thus seems reasonable to assume that the perturbation pressure is determined by some combination of these other modes and the sound mode. This point will be investigated further in the next section.

Figures 4 and 5 are a comparison of the $n = 8$ discrete ordinate solution with the polynomial expansion theories. All of the theories except the one of Kahn and Mintzer [7] are based upon a perturbation around a Maxwellian equilibrium distribution. The discrete ordinate solution compares favorably with the expansion theories. Note that the $n = 8$ discrete ordinate solution gives results which are comparable to those of Pekeris, et al. [5], for a polynomial expansion in which twenty-six terms are retained. The $n = 8$ solution is also very close to the Grad 25 moment approximation [7]. The expansion theories all converge to the discrete ordinate solution for the CT model at large values of $r(r > 3)$.

A comparison of the discrete ordinate $n = 8$ solution with the transformation theories is given in Figures 6 and 7. The discrete ordinate solution is seen to be in good agreement with these theories for values of r for which convergence was obtained.

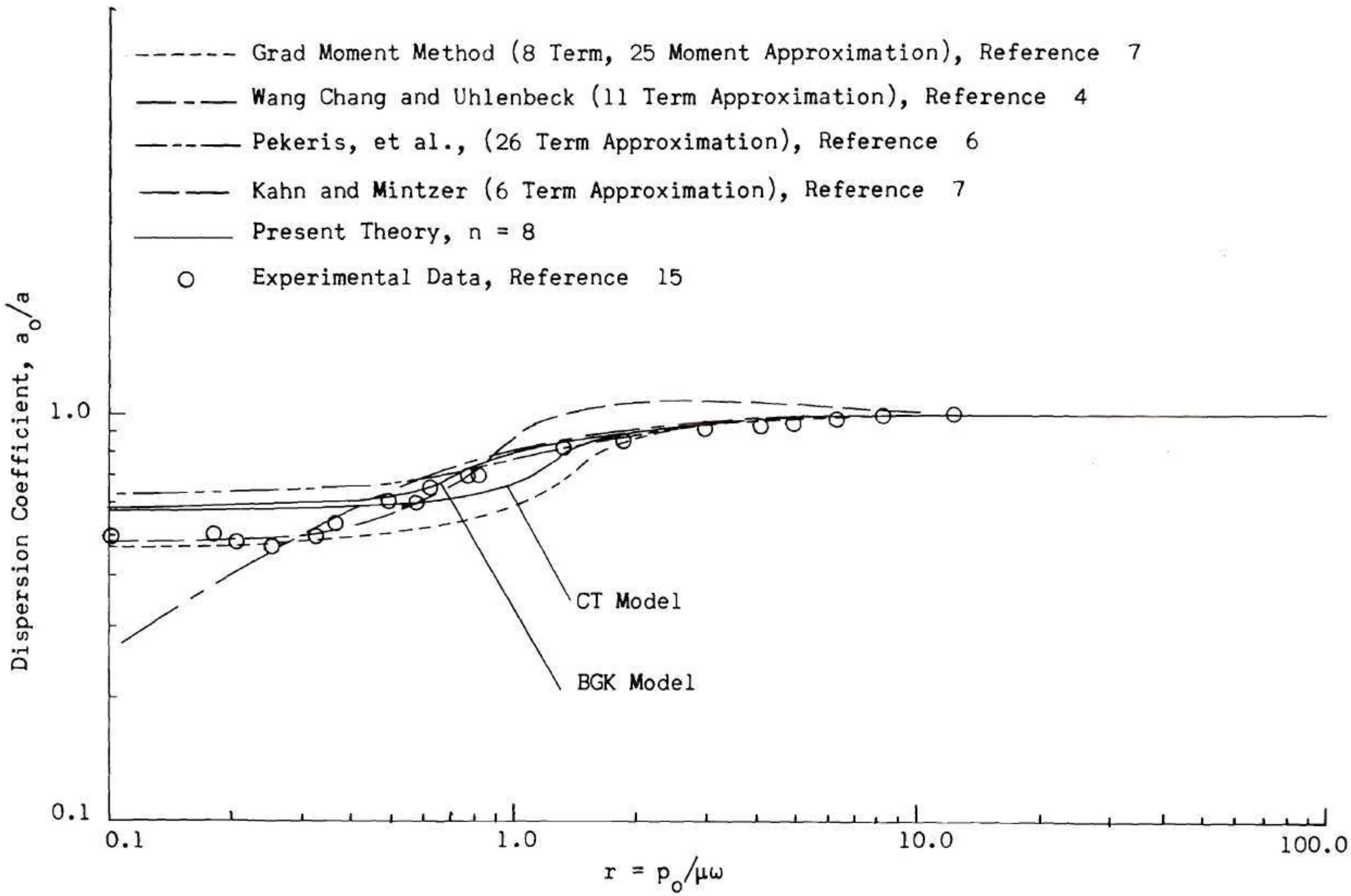

Figure 4. Comparison of Discrete Ordinate n = 8 Solution for the Dispersion Coefficient With Polynomial Expansion Theories.

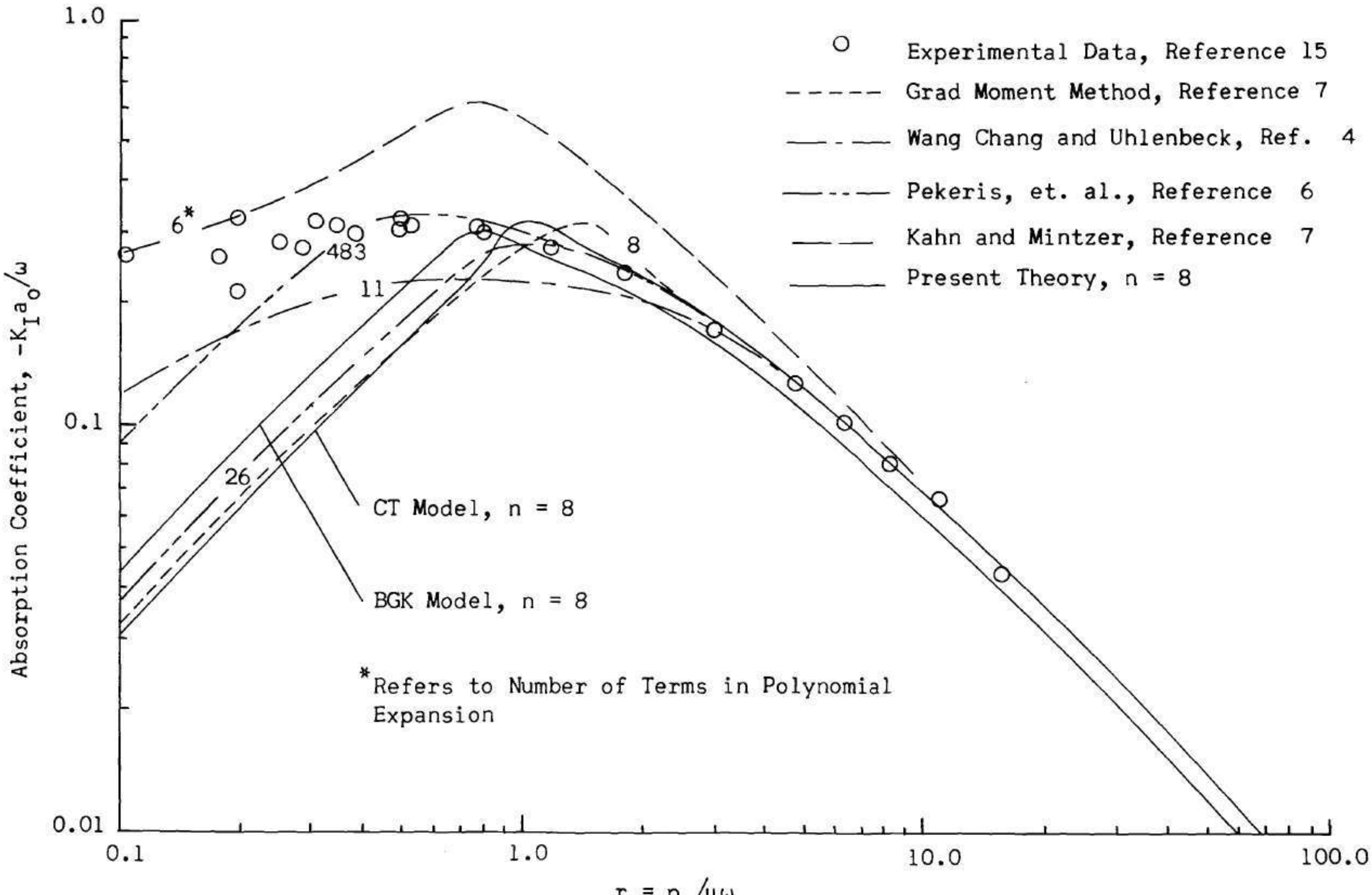

Figure 5. Comparison of the Discrete Ordinate n = 8 Solution for the Absorption Coefficient With Polynomial Expansion Theories.

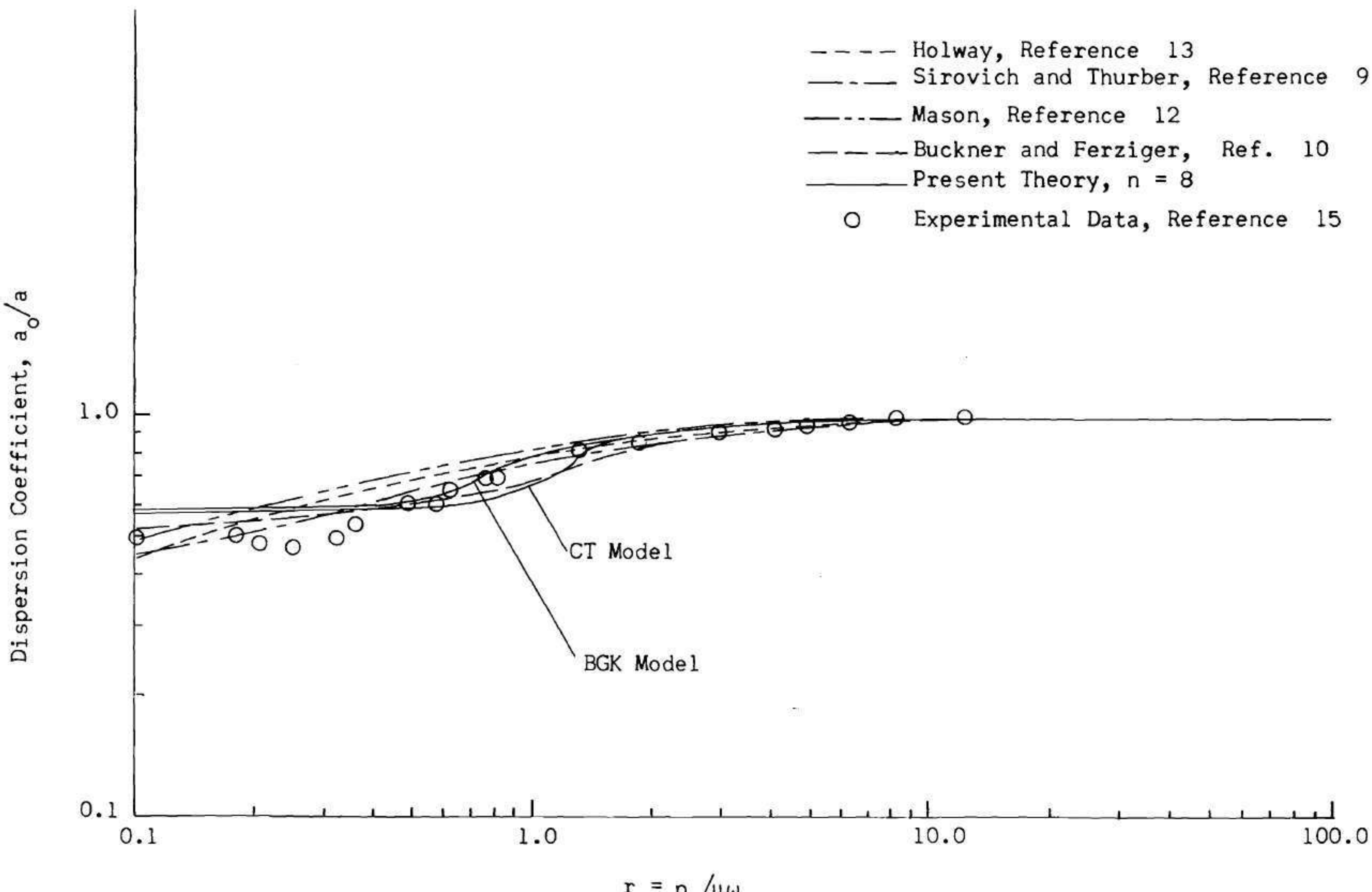

Figure 6. Comparison of Discrete Ordinate n = 8 Solution for the Dispersion Coefficient With Transformation Theories.

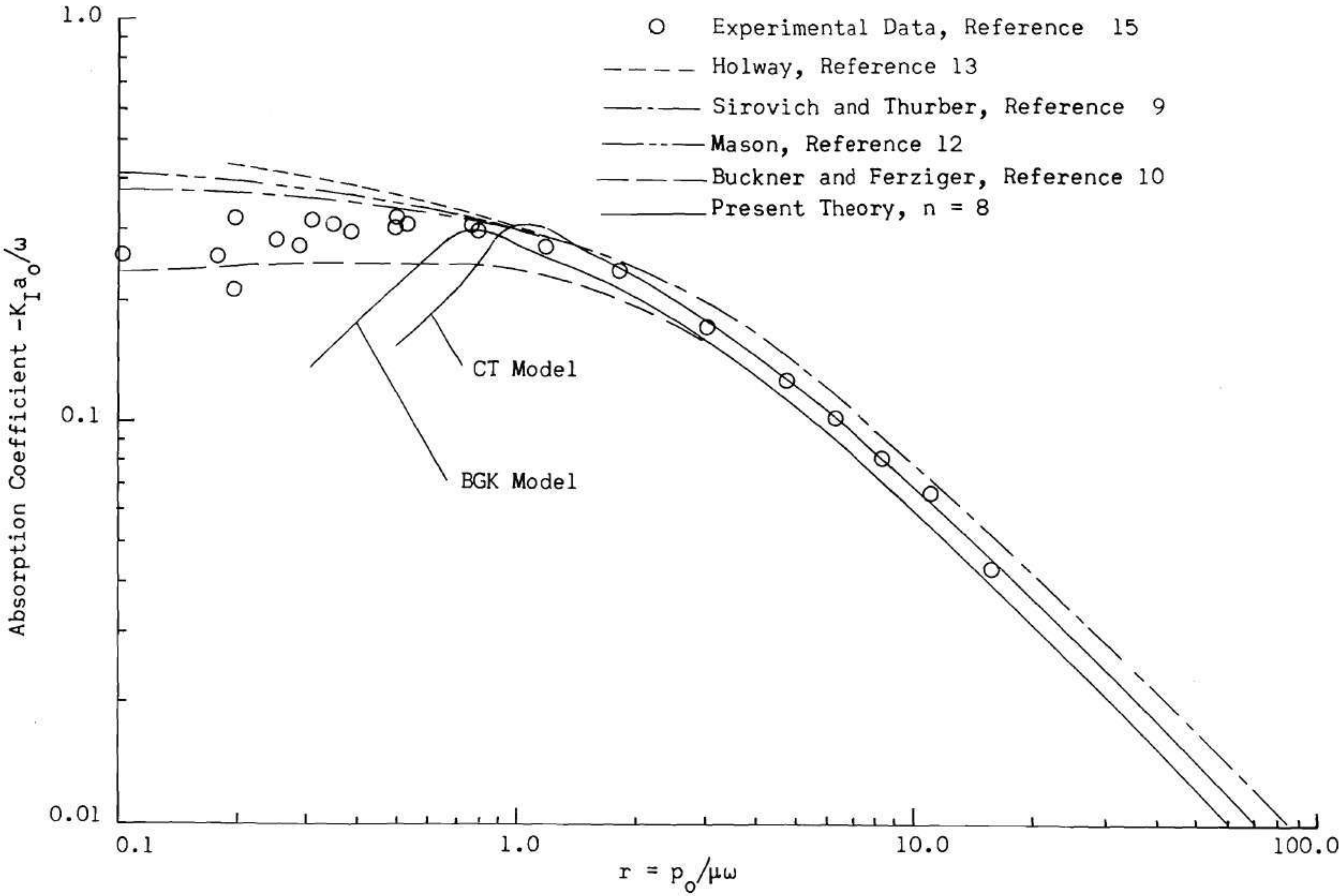

Figure 7. Comparison of Discrete Ordinate n = 8 Solution for the Absorption Coefficient With Transformation Theories.

Theoretical Formulation (Highly Rarefied Solution)

BGK Model

Discussion of the Model Equation. In this section, a solution valid in a highly rarefied gas is obtained by numerically solving the BGK model with appropriate boundary conditions. By using this method of solution, one may calculate the perturbation pressure field. An analysis of this field will then yield the absorption and dispersion of the wave.

The governing equation is the linearized BGK model [11]

$$\Theta\beta_o \frac{\partial\Phi}{\partial t} + \Theta c_x \frac{\partial\Phi}{\partial x} + \Phi = \pi^{-3/2} \int e^{-c_1^2} \Phi(x, \bar{c}_1, t) d^3c_1 \tag{41}$$

$$+ 2c_x \pi^{-3/2} \int e^{-c_1^2} c_{x_1} \Phi(x, \bar{c}_1, t) d^3c_1$$

$$+ \frac{2}{3}\left(c^2 - \frac{3}{2}\right)\pi^{-3/2} \int e^{-c_1^2}\left(c_1^2 - \frac{3}{2}\right)\Phi(x, \bar{c}_1, t) d^3c_1 \; .$$

Initially, $\Phi(x, \bar{c}, 0) \equiv 0$. The boundary conditions at the piston are found by assuming that the molecules leaving the piston do so with a Maxwellian distribution characteristic of the motion of the piston (see Reference [7]). Linearizing this distribution function yields

$$\Phi(x = 0, \bar{c}, t) = 2c_x u_o \sin \omega t \; , \quad c_x > 0 \tag{42}$$

$$= 0 \qquad , \quad c_x < 0 \tag{43}$$

where u_o is the maximum velocity of the piston nondimensionalized by $1/\beta_o$. Physically, Equations (42) and (43) state that only the molecules moving away from the piston are perturbed from a Maxwellian distribution. It is thus necessary to solve only for $\Phi(x, c_x > 0, t)$.

Transformation of the Integral Equation. For reasons discussed earlier, it is desirable to eliminate the (c_y, c_z) integrations from the governing equation. This may again be accomplished by introducing the functions ψ and η, except that now ψ and η are space and time dependent, i.e.,

$$\psi(x, c_x, t) = \pi^{-3/2} \int_{-\infty}^{+\infty} dc_y \int_{-\infty}^{+\infty} dc_z e^{-c_y^2 - c_z^2} \Phi(x, \bar{c}, t) \tag{44}$$

$$\eta(x, c_x, t) = \pi^{-3/2} \int_{-\infty}^{+\infty} dc_y \int_{-\infty}^{+\infty} dc_z e^{-c_y^2 - c_z^2} (c_y^2 + c_z^2) \Phi(x, \bar{c}, t) \tag{45}$$

Introducing Equations (44) and (45) into Equation (41) and performing suitable integrations yield

$$\begin{aligned} \sqrt{\pi}\left[\Theta\beta_o \frac{\partial\psi}{\partial t} + c_x \Theta \frac{\partial\psi}{\partial x} + \psi\right] &= \int_{-\infty}^{+\infty} e^{-c_{x_1}^2} \psi(x, c_{x_1}, t) dc_{x_1} \\ &+ 2c_x \int_{-\infty}^{+\infty} e^{-c_{x_1}^2} c_{x_1} \psi(x, c_{x_1}, t) dc_{x_1} \\ &+ \frac{2}{3}\left(c_x^2 - \frac{1}{2}\right) \int_{-\infty}^{+\infty} e^{-c_{x_1}^2} \left[\eta(x, c_{x_1}, t) + \left(c_{x_1}^2 - \frac{3}{2}\right)\psi(x, c_{x_1}, t)\right] dc_{x_1} \end{aligned} \tag{46}$$

and

$$\sqrt{\pi}\left[\Theta\beta_o \frac{\partial\eta}{\partial t} + c_x\Theta \frac{\partial\eta}{\partial x} + \eta\right] = \int_{-\infty}^{+\infty} e^{-c_{x_1}^2} \psi(x, c_{x_1}, t)dc_{x_1} \tag{47}$$

$$+ 2c_x \int_{-\infty}^{+\infty} e^{-c_{x_1}^2} c_{x_1}\psi(x, c_{x_1}, t)dc_{x_1}$$

$$+ \frac{2}{3}(c_x^2 + \frac{1}{2}) \int_{-\infty}^{+\infty} e^{-c_{x_1}^2} \Big[\eta(x, c_{x_1}, t)$$

$$+ (c_{x_1}^2 - \frac{3}{2})\psi(x, c_{x_1}, t)\Big] dc_{x_1} \quad .$$

The transformed boundary conditions are

$$\begin{aligned} \psi(x = 0, c_x, t) &= 2(\pi)^{-1/2} u_o c_x \sin \omega t \quad , \qquad c_x > 0 \\ \eta(x = 0, c_x, t) &= 2(\pi)^{-1/2} u_o c_x \sin \omega t \quad , \qquad c_x > 0 \end{aligned} \tag{48}$$

<u>Transformation of Infinite Interval to Finite Interval</u>. Before approximating Equations (46) and (47) by difference equations it is desirable to transform the interval $0 \leq x \leq \infty$ into the interval $0 \leq \xi \leq 1$. This is readily accomplished by use of the transformation [40]

$$\xi = \tanh \varepsilon x \ , \tag{49}$$

where ε is a parameter that may be varied to alter the spacing of grid points in x-space. Using this transformation, one may show that the derivative with respect to x becomes

$$\frac{\partial}{\partial x} = \varepsilon(1 - \xi^2)\frac{\partial}{\partial \xi} \quad . \tag{50}$$

Discrete Ordinate Approximation. By approximating the integrals in Equations (46) and (47) by finite summations and formally performing the interval transformation, one obtains the coupled system of differential equations given below:

$$\sqrt{\pi}\left(\frac{16}{5\pi r}\frac{\partial\psi_\ell}{\partial\tau} + \varepsilon^* c_\ell (1 - \xi^2)\frac{\partial\psi_\ell}{\partial\xi} + \psi_\ell\right) = \tag{51}$$

$$\sum_{i=n+1}^{2n} H_i\left[1 + 2c_\ell c_i + \frac{2}{3}\left(c_\ell^2 - \frac{1}{2}\right)\left(c_i^2 - \frac{3}{2}\right)\right]\psi_i$$

$$+ \sum_{i=n+1}^{2n} H_i \frac{2}{3}\left(c_\ell^2 - \frac{1}{2}\right)\eta_i$$

$$\sqrt{\pi}\left(\frac{16}{5\pi r}\frac{\partial\eta_\ell}{\partial\tau} + \varepsilon^* c_\ell (1 - \xi^2)\frac{\partial\eta_\ell}{\partial\xi} + \eta_\ell\right) = \tag{52}$$

$$\sum_{i=n+1}^{2n} H_i\left[1 + 2c_\ell c_i + \frac{2}{3}\left(c_\ell^2 + \frac{1}{2}\right)\left(c_i^2 - \frac{3}{2}\right)\right]\psi_i$$

$$+ \sum_{i=n+1}^{2n} H_i \frac{2}{3}\left(c_\ell^2 + \frac{1}{2}\right)\eta_i$$

where

$$\ell = n+1, \ldots, 2n, \ \tau = \omega t, \ \varepsilon^* = \Theta\varepsilon \text{ and } \frac{16}{5\pi r} = \omega\Theta\beta_o \quad .$$

Note that the summation limits reflect the fact that the molecules approaching the piston are not perturbed from a Maxwellian distribution for this highly rarefied case.

The boundary conditions are:

$$\psi_\ell(\xi = 0, c_\ell, \tfrac{\tau}{\omega}) = 2(\pi)^{-1/2} u_o c_\ell \sin\tau \tag{53}$$

$$\eta_\ell(\xi = 0, c_\ell, \tfrac{\tau}{\omega}) = 2(\pi)^{-1/2} u_o c_\ell \sin\tau \tag{54}$$

Finite Difference Approximation. The lattice shown in Figure 8 is used in the finite difference approximation. Steps in time are $\Delta\tau$ apart, and the interval $0 \le \xi \le 1$ is subdivided into M equal intervals of length $\Delta\xi$. The derivative in time is approximated by a backwards difference scheme, and the ξ-derivative by a central difference. Thus at the j^{th} station at time τ the derivatives are approximated by

$$\left.\begin{aligned} \frac{\partial\psi_\ell}{\partial\tau} &\cong \frac{\psi_\ell(j,\tau) - \psi_\ell(j, \tau - \Delta\tau)}{\Delta\tau} \\ \frac{\partial\eta_\ell}{\partial\tau} &\cong \frac{\eta_\ell(j, \tau) - \eta_\ell(j, \tau - \Delta\tau)}{\Delta\tau} \end{aligned}\right\} \tag{57}$$

$$(1 - \xi^2)\frac{\partial\psi_\ell}{\partial\xi} \cong (1 - \xi^2(j))\, \frac{\psi_\ell(j+1,\tau)-\psi_\ell(j-1,\tau)}{2\Delta\xi}$$

and

$$(1 - \xi^2)\frac{\partial\eta_\ell}{\partial\xi} \cong (1 - \xi^2(j))\, \frac{\eta_\ell(j+1,\tau)-\eta_\ell(j-1,\tau)}{2\Delta\xi} \tag{58}$$

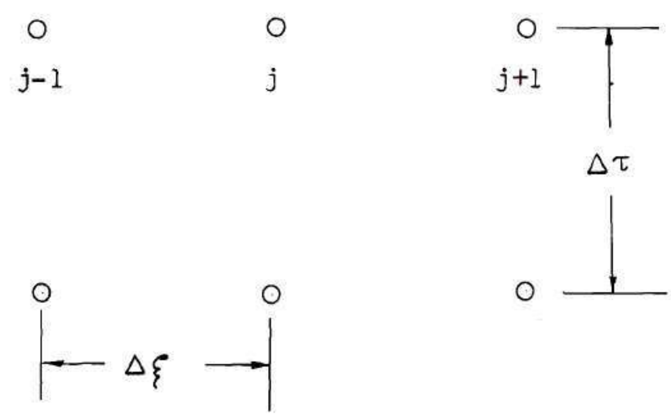

Figure 8. Grid for Highly Rarefied Solution.

By introducing Equations (57) and (58) into Equations (51) and (52), one obtains

$$
\begin{aligned}
&- \frac{c_\ell \varepsilon^*(1 - \xi^2(j))}{2\Delta\xi} \psi_\ell^{\hat{n}+1}(j - 1) + \left[\frac{16}{5\pi r \Delta\tau} + 1\right] \psi_\ell^{\hat{n}+1}(j) \\
&+ \frac{c_\ell \varepsilon^*(1 - \xi^2(j))}{2\Delta\xi} \psi_\ell^{\hat{n}+1}(j + 1) = \frac{16}{5\pi r \Delta\tau} \psi_\ell(j, \tau - \Delta\tau) \\
&+ \frac{1}{\sqrt{\pi}} \sum_{i=n+1}^{2n} H_i \left[1 + 2c_\ell c_i + \frac{2}{3}(c_\ell^2 - \frac{1}{2})(c_i^2 - \frac{3}{2})\right] \psi_i^{\hat{n}}(j) \\
&+ \frac{1}{\sqrt{\pi}} \sum_{i=n+1}^{2n} H_i \left[\frac{2}{3}(c_\ell^2 - \frac{1}{2})\right] \eta_i^{\hat{n}}(j)
\end{aligned}
\tag{59}
$$

and

$$
\begin{aligned}
&- \frac{c_\ell \varepsilon^*(1 - \xi^2(j))}{2\Delta\xi} \eta_\ell^{\hat{n}+1}(j - 1) + \left[\frac{16}{5\pi \Delta\tau r} + 1\right] \eta_\ell^{\hat{n}+1}(j) \\
&+ \frac{c_\ell \varepsilon^*(1 - \xi^2(j))}{2\Delta\xi} \eta_\ell^{\hat{n}+1}(j+1) = \frac{16}{5\pi r \Delta\tau} \eta_\ell(j, \tau - \Delta\tau) \\
&+ \frac{1}{\sqrt{\pi}} \sum_{i=n+1}^{2n} H_i \left[1 + 2c_\ell c_i + \frac{2}{3}(c_\ell^2 + \frac{1}{2})(c_i^2 - \frac{3}{2})\right] \psi_i^{\hat{n}}(j) \\
&+ \frac{1}{\sqrt{\pi}} \sum_{i=n+1}^{2n} H_i \left[\frac{2}{3}(c_\ell^2 + \frac{1}{2})\right] \eta_i^{\hat{n}}(j) ,
\end{aligned}
\tag{60}
$$

where

$\ell = n+1, n+2, \ldots, 2n, \quad j = 1, 2, \ldots, M-1$ and $\hat{n}+1$ denotes the $(\hat{n} + 1)^{th}$ approximation to the solution at time τ.

Equations (59) and (60) may be solved simultaneously to yield the functions $\psi_\ell(j)$ and $\eta_\ell(j)$ for $\ell = n+1, n+2, \ldots, 2n, j=1, 2, \ldots, M-1$. From these two functions it is a simple matter to calculate the perturbation pressure field resulting from the oscillating piston. An analysis of the pressure field will yield the phase velocity and absorption coefficient of the wave. The method by which the absorption and dispersion are calculated is discussed in the section on the computational procedure.

Computational Procedure (Highly Rarefied Solution)

The system of equations represented by Equations (59) and (60) is tridiagonal and may be readily solved to yield the $(\hat{n} + 1)^{th}$ approximation to the solution at time τ. Once the functions ψ_ℓ and η_ℓ are known for a given time τ, the perturbation pressure p'_{xx} at station j may be found from

$$p'_{xx}(\tau,j) = \sum_{i=n+1}^{2n} H_i\left[c_i^2\psi_i(\tau,j)\right] \quad . \tag{61}$$

It is the analysis of this pressure field which yields the absorption and dispersion coefficients.

In a highly rarefied gas the absorption and dispersion coefficients are functions of position. The pertinent parameter is S, which in the present notation may be expressed as

$$S = \sqrt{2}\,\omega\beta_o x \quad . \tag{62}$$

For the experiments of Meyer and Sessler [15], S = 12.15. Since

$\omega\Theta\beta_o = \frac{16}{5\pi}\frac{1}{r}$, one may conclude that

$$\frac{x}{\ell} = \sqrt{\frac{2}{\pi}}\,\frac{5\pi}{16}\, rS \quad , \tag{63}$$

where ℓ is the mean free path.

The term $\frac{x}{\ell}$ is related to ξ by the inverse of the interval transformation [40]:

$$\frac{x}{\ell} = \frac{1}{\sqrt{\pi}\,\varepsilon^*} \ln \frac{1+\xi}{1-\xi} \tag{64}$$

Equations (63) and (64) thus specify the value of ξ at which the pressure field is to be analyzed.

The absorption coefficient is determined from the theoretically calculated pressure field exactly as it is obtained from the experimental pressure field, by computing the logarithmic decrement of the pressure amplitude. The phase velocity may be found from the progression of the wave during the time τ. A sample calculation for a particular point is given in the Appendix.

Discussion of Results (Highly Rarefied Solution)

In this section the results obtained in numerically solving the linearized Boltzmann equation with the BGK model are compared with other theories and with the experimental data of Meyer and Sessler [15]. For these calculations the interval was divided into 150 equal segments; the time step was $\Delta\tau = \frac{2\pi}{10}$. For r of order 0.01, $\varepsilon^* = 1.0$ was sufficient. For r of order 0.1, it was necessary to use $\varepsilon^* = 0.5$ (this essentially moves the grid points further from the piston in x-space, and is necessary because of the decreased damping which occurs as r increases). The non-dimensional velocity of the piston, u_o, was assumed to be 10^{-3}. All calculations are for n = 5.

The numerical scheme converged to a solution very rapidly for r = 0.01 (approximately 2 iterations were required). For larger values of r ($r \approx 0.1$), more iterations were necessary because the wave propagated further from the piston. A few calculations were performed with 200 grid points, and the solution obtained agreed with the M = 150 solution to within a few per cent.

This solution technique is capable of predicting the transient development of the pressure field. Figures 9 through 11 show the development of the pressure field for r = 0.01. In these figures the perturbation pressure, p'_{xx}, is plotted versus ξ. Figure 9 is for the first oscillation of the piston ($0 < \tau \leq 2\pi$) and reflects the fact that the piston initially moves forward. By the time $\tau = 2\pi$, the expansion wave generated by the rearward motion of the piston is beginning to influence the pressure field. The pressure field for $4\pi < \tau \leq 6\pi$ is shown in Figure 10. The pressure field is approaching a steady state, however, the influence of the initial compression wave is still apparent. Figure 11 is for $8\pi < \tau \leq 10\pi$. Note that the envelope of the pressure field is nearly symmetric. The steady state solution is thus obtained after approximately 5 oscillations of the piston.

The solution* presented in Figure 12 is for S = 12.15, which corresponds to the experimental data obtained by Meyer and Sessler [15]. The absorption predicted by the discrete ordinate solution is in very good

*The details of the calculation for the absorption and dispersion are given in the Appendix.

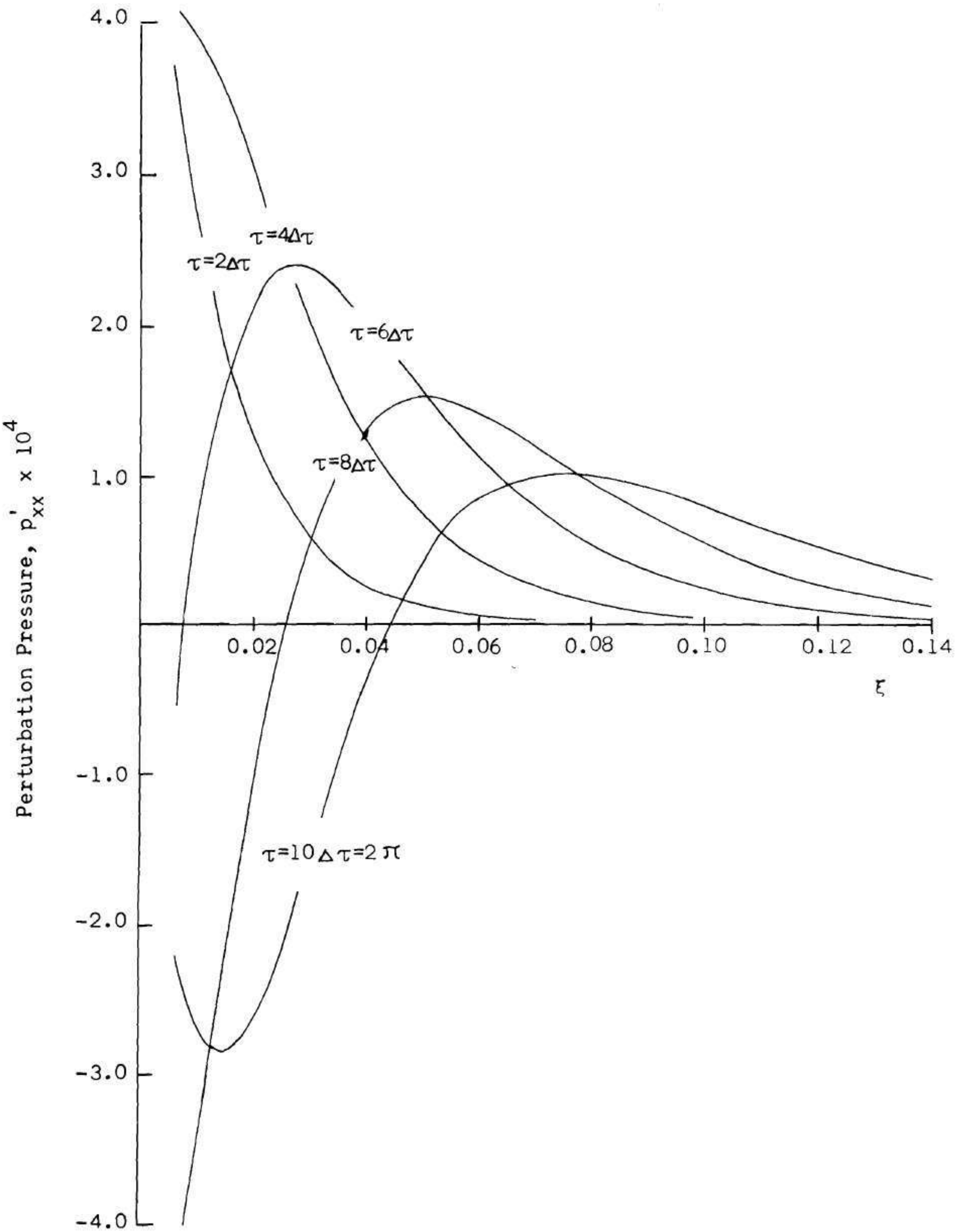

Figure 9. Perturbation Pressure Field for $r = 0.01$, $0 < \tau \leq 2\pi$.

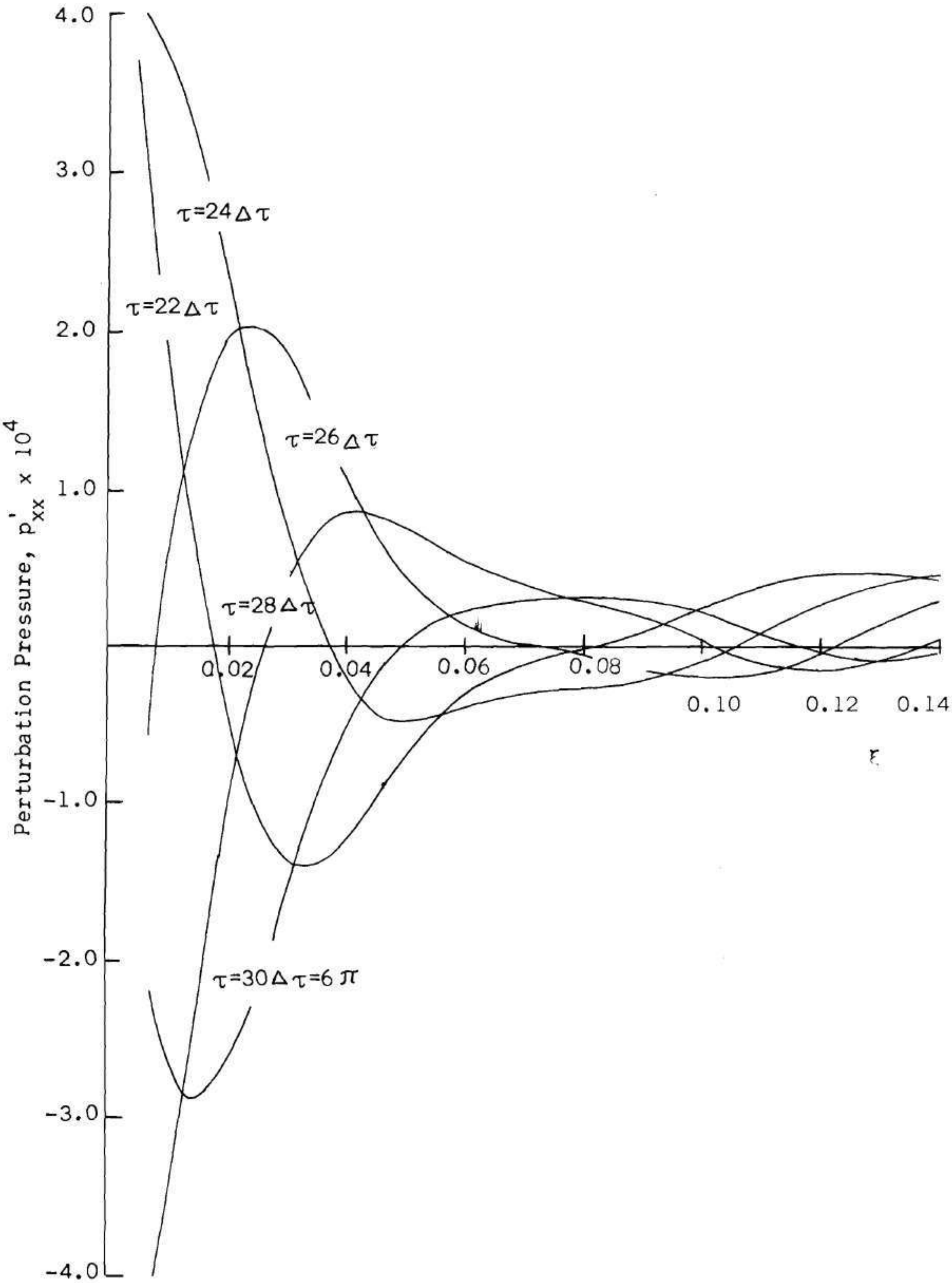

Figure 10. Perturbation Pressure Field for r = 0.01, $4\pi < \tau \leq 6\pi$.

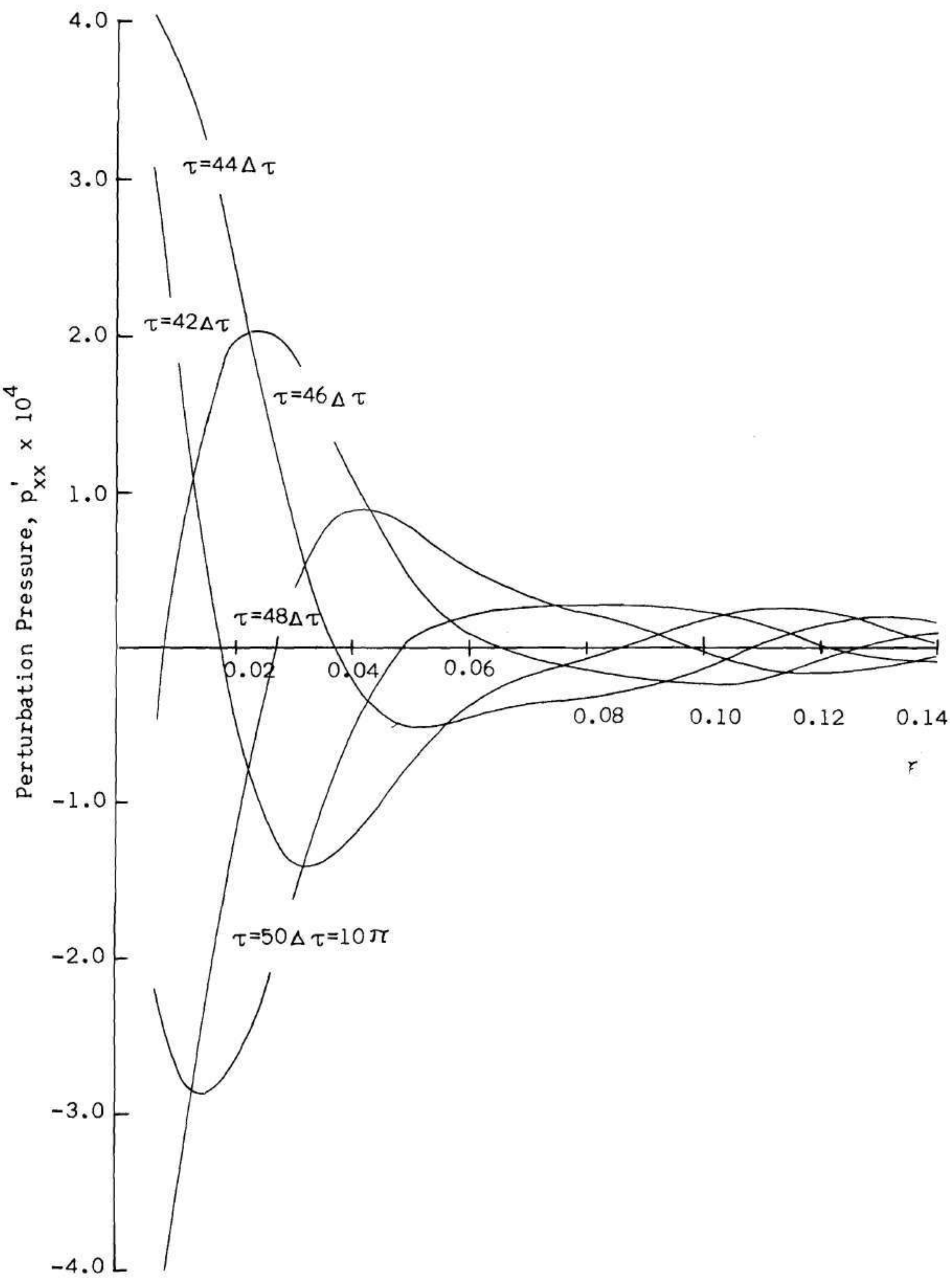

Figure 11. Perturbation Pressure Field for r = 0.01, $8\pi < \tau \leq 10\pi$.

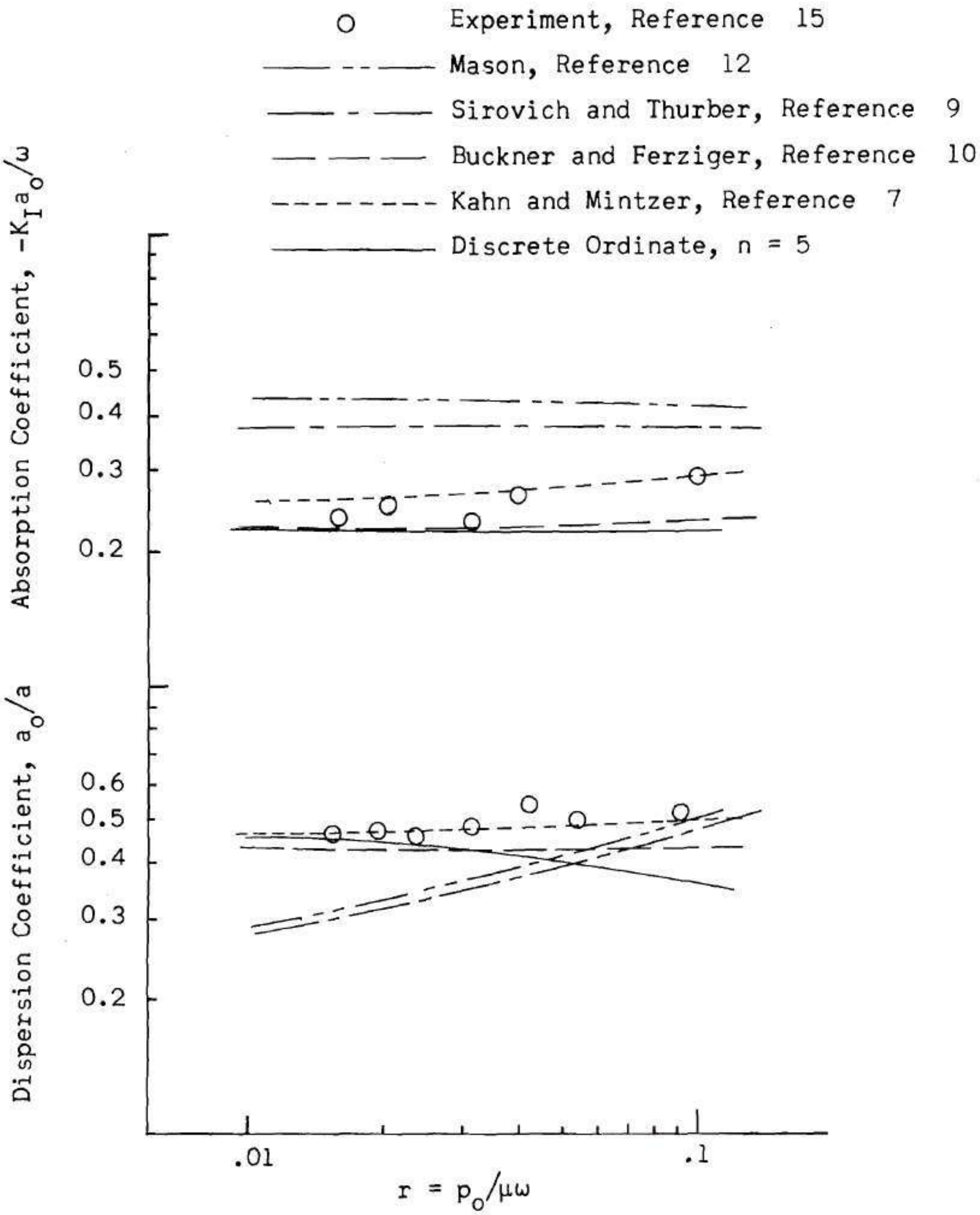

Figure 12. Comparison of Highly Rarefied Solution With Theory and Experiment.

agreement with the experimental data. The dispersion is predicted correctly at $r = 0.01$, but deviates from the experimental data and the other theories for $r = 0.1$. Solutions for larger values of r were not calculated because it became increasingly difficult to obtain convergence at a given time τ. This is due to the fact that as r increases, the wave propagates further from the piston.

CHAPTER III

FREE SOUND WAVE PROPAGATION IN A RAREFIED GAS

Theoretical Formulation

Two different physical situations arise when one is considering wave propagation in a neutral gas. The analysis of the preceding chapter considered forced wave propagation - the wave motion resulting from an oscillating piston. In this chapter the propagation of an initial disturbance (free sound propagation) will be investigated. An infinite volume of gas of mass m, temperature T_o and number density n_o is perturbed away from equilibrium at time t = 0. The perturbation is harmonic and is a function of x. The problem is to determine the subsequent evolution of this initial disturbance. The analysis will again be based upon the model equations proposed by Bhatnagar, Gross and Krook [11] and by Cercignani and Tironi [33].

BGK Model

The governing equation is the linearized BGK model

$$\beta_o \frac{\partial \Phi}{\partial t} + c_x \frac{\partial \Phi}{\partial x} = \frac{1}{\Theta}\Big[- \Phi + \pi^{-3/2} \int e^{-c_1^2} \Phi(x, \bar{c}_1, t) d^3 c_1 \tag{1}$$

$$+ 2 c_x \pi^{-3/2} \int e^{-c_1^2} c_{x_1} \Phi(x, \bar{c}_1, t) d^3 c_1$$

$$+ \frac{2}{3}(c^2 - \frac{3}{2}) \pi^{-3/2} \int e^{-c_1^2} (c_1^2 - \frac{3}{2}) \Phi(x, \bar{c}_1, t) d^3 c_1 \Big] ,$$

which was discussed in detail in the previous chapter.

For forced sound propagation the problem was to determine the wave number of the distrubance for a given sound frequency and molecular collision frequency. Here, one must compute the frequency of the disturbance for a given initial wave number and molecular collision frequency. Accordingly, the perturbation $\Phi(x, \bar{c}, t)$ is assumed to be of the form

$$\Phi(x, \bar{c}, t) = \varphi(\bar{c})e^{-\sigma t+\hat{i}Kx} \tag{2}$$

The absorption of the wave will be given by σ_R, and the phase velocity by $\frac{\sigma_I}{K}$, where σ_R and σ_I are the real and imaginary parts of σ, respectively. K is the wave number of the initial disturbance ($K = 2\pi/\Lambda$, where Λ is the wavelength).

By substituting Equation (2) into Equation (1) and transforming the resultant integral equation for $\varphi(\bar{c})$, one obtains

$$\sqrt{\pi}\left(-\frac{16}{5\pi}\frac{1}{r}\hat{\sigma} + \hat{i}\,\frac{16}{5\pi}\sqrt{6/5}\,\frac{1}{r}\,c_x + 1\right)\psi(c_x) = \tag{3}$$

$$\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\psi(c_{x_1})dc_{x_1} + 2c_x\int_{-\infty}^{+\infty} e^{-c_{x_1}^2} c_{x_1}\psi(c_{x_1})dc_{x_1}$$

$$+ \frac{2}{3}(c_x^2 - \frac{1}{2})\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\left[\eta(c_{x_1}^2) + (c_{x_1}^2 - \frac{3}{2})\psi(c_{x_1})\right]dc_{x_1}$$

and

$$\sqrt{\pi}\left(-\frac{16}{5\pi}\frac{1}{r}\hat{\sigma} + \hat{i}\,\frac{16}{5\pi}\sqrt{6/5}\,\frac{1}{r}\,c_x + 1\right)\eta(c_x) = \tag{4}$$

$$\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\psi(c_{x_1})dc_{x_1} + 2c_x\int_{-\infty}^{+\infty} e^{-c_{x_1}^2} c_{x_1}\psi(c_{x_1})dc_{x_1}$$

$$+ \frac{2}{3}(c_x^2 + \frac{1}{2}) \int_{-\infty}^{+\infty} e^{-c_{x_1}^2} \left[\eta(c_{x_1}) + (c_{x_1}^2 - \frac{3}{2})\phi(c_{x_1}) \right] dc_{x_1} \ ,$$

where

$$\hat{\sigma} = \frac{\sigma}{\omega_o} \ , \tag{5}$$

$$r = \frac{p_o}{\mu\omega_o} \tag{6}$$

and

$$\omega_o = Ka_o \ . \tag{7}$$

By approximating the integrals in Equations (3) and (4) by finite summations, one obtains a system of equations of the form

$$\bar{\bar{A}} \begin{pmatrix} \bar{\phi} \\ - \\ \bar{\eta} \end{pmatrix} = \hat{\sigma}\bar{\bar{I}} \begin{pmatrix} \bar{\phi} \\ - \\ \bar{\eta} \end{pmatrix} \ , \tag{8}$$

where

$$A_{\ell i} = (\hat{i}\ \sqrt{6/5}\ c_\ell + \frac{5\pi}{16} r)\delta_{\ell i} - \frac{5\sqrt{\pi}}{16} rH_i \Big[1 + 2c_\ell c_i + \frac{2}{3}(c_\ell^2 - \frac{1}{2})(c_i^2 - \frac{3}{2}) \Big] \ , \tag{9}$$

$$A_{\ell, i+2n} = - \frac{5\sqrt{\pi}}{16} rH_i \Big[\frac{2}{3}(c_\ell^2 - \frac{1}{2}) \Big] \ , \tag{10}$$

$$A_{\ell+2n, i} = - \frac{5\sqrt{\pi}}{16} rH_i \Big[1 + 2c_\ell c_i + \frac{2}{3}(c_\ell^2 + \frac{1}{2})(c_i^2 - \frac{3}{2}) \Big] \ , \tag{11}$$

and

$$A_{\ell+2n,i+2n} = (\hat{i}\sqrt{6/5}\, c_\ell + \frac{5\pi}{16} r)\delta_{\ell i} - \frac{5\sqrt{\pi}}{16} rH_i\left[\frac{2}{3}(c_\ell^2 + \frac{1}{2})\right]. \tag{12}$$

Equation (8) will have a nontrivial solution only if $\hat{\sigma}$ is an eigenvalue of $\bar{\bar{A}}$.

CT Model

The analysis of this section is based upon the CT model of the Boltzmann equation (Equation (33) of the previous chapter). By substituting Equation (2) into the CT model and transforming the resultant equation, one obtains

$$\sqrt{\pi}(-\frac{8(\lambda + 2)}{5\pi r}\hat{\sigma} + \hat{i}\,\frac{8(\lambda + 2)}{5\,\pi}\sqrt{6/5}\,\frac{1}{r}\,c_x + 1)\psi(c_x) = \tag{13}$$

$$\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\psi(c_{x_1})dc_{x_1} + 2c_x\int_{-\infty}^{+\infty} e^{-c_{x_1}^2} c_{x_1}\psi(c_{x_1})dc_{x_1}$$

$$+ \frac{2}{3}(c_x^2 - \frac{1}{2})\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\left[\eta(c_{x_1}) + (c_{x_1}^2 - \frac{3}{2})\psi(c_{x_1})\right]dc_{x_1}$$

$$- \lambda\left[c_x^2\int_{-\infty}^{+\infty} e^{-c_{x_1}^2} c_{x_1}^2\psi(c_{x_1})dc_{x_1} + \frac{1}{2}\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\eta(c_{x_1})dc_{x_1}\right]$$

$$+ \frac{\lambda}{3}(c_x^2 + 1)\left[\int_{-\infty}^{+\infty} e^{-c_{x_1}^2} c_{x_1}^2\psi(c_{x_1})dc_{x_1} + \int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\eta(c_{x_1})dc_{x_1}\right]$$

and

$$\sqrt{\pi}\left(-\frac{8(\lambda+2)}{5\pi r}\hat{\sigma}+\hat{i}\,\frac{8(\lambda+2)}{5\pi}\sqrt{6/5}\,\frac{1}{r}\,c_x+1\right)\eta(c_x)= \tag{14}$$

$$\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\psi(c_{x_1})dc_{x_1}+2c_x\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}c_{x_1}\psi(c_{x_1})dc_{x_1}$$

$$+\frac{2}{3}(c_x^2+\frac{1}{2})\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\left[\eta(c_{x_1})+(c_{x_1}^2-\frac{3}{2})\psi(c_{x_1})\right]dc_{x_1}$$

$$-\lambda\left[c_x^2\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}c_{x_1}^2\psi(c_{x_1})dc_{x_1}+\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\eta(c_{x_1})dc_{x_1}\right]$$

$$+\frac{\lambda}{3}(c_x^2+2)\left[\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}c_{x_1}^2\psi(c_{x_1})dc_{x_1}+\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\eta(c_{x_1})dc_{x_1}\right],$$

where $\hat{\sigma}$ and r are defined by Equations (5) and (6), respectively.

Applying the discrete ordinate to Equations (13) and (14) yields a system of the form

$$\bar{\bar{A}}\begin{pmatrix}\bar{\psi}\\ \bar{\eta}\end{pmatrix}=\hat{\sigma}\bar{\bar{I}}\begin{pmatrix}\bar{\psi}\\ \bar{\eta}\end{pmatrix}.$$

Here, $\bar{\bar{A}}$ has the elements

$$A_{\ell i}=\left(\hat{i}\sqrt{6/5}\,c_\ell+\frac{5\pi r}{8(\lambda+2)}\right)\delta_{\ell i}-\frac{5\sqrt{\pi}\,r}{8(\lambda+2)}H_i\Big[1+2c_\ell c_i \tag{15}$$

$$+\frac{2}{3}(c_\ell^2-\frac{1}{2})(c_i^2-\frac{3}{2})-\frac{2}{3}\lambda(c_\ell^2-\frac{1}{2})c_i^2\Big]$$

$$A_{\ell,i+2n} = -\frac{5\sqrt{\pi}\,r}{8(\lambda+2)} H_i\left[\frac{2}{3}(c_\ell^2 - \frac{1}{2}) + \lambda(\frac{c_\ell^2}{3} - \frac{1}{6})\right] \tag{16}$$

$$A_{\ell+2n,i} = -\frac{5\sqrt{\pi}\,r}{8(\lambda+2)} H_i\left[1 + 2c_\ell c_i + \frac{2}{3}(c_\ell^2 + \frac{1}{2})(c_i^2 - \frac{3}{2}) - \frac{2}{3}\lambda(c_\ell^2 - 1)c_i^2\right] \tag{17}$$

$$A_{\ell+2n,i+2n} = (\hat{\imath}\sqrt{6/5}\,c_\ell + \frac{5\pi r}{8(\lambda+2)})\delta_{\ell i} - \frac{5\sqrt{\pi}\,r}{8(\lambda+2)} \cdot H_i\left[\frac{2}{3}(c_\ell^2 + \frac{1}{2}) + \lambda(\frac{1}{3}c_\ell^2 - \frac{1}{3})\right], \tag{18}$$

where

$\ell = 1, 2, \ldots, 2n, \quad i = 1, 2, \ldots, 2n.$

As before, this system of algebraic equations has a nontrivial solution only if $\hat{\sigma}$ is an eigenvalue of $\bar{\bar{A}}$. Once the eigenvalues are computed, one may find the absorption and dispersion of the wave from the real and imaginary parts of $\hat{\sigma}$, respectively.

Discussion of Results

The eigenvalues of $\bar{\bar{A}}$ for both the BGK and CT model were computed for $n = 8$ for $0.1 \leq r < 100$. A few calculations for $n = 6$ and 7 indicated that the range of convergence could be extended by increasing the value of n, however, as in the case of forced sound waves, the extension is slight (the point at which the $n = 7$ solution deviated from the $n = 8$ solution is indicated). All calculations for the CT model are for $\lambda = 1.0$.

The $n = 8$ discrete ordinate solutions are compared with the theories presented in Reference [16] in Figures 13 and 14. In these figures, the dispersion coefficient, $\frac{a}{a_o}$, and the absorption coefficient, $\frac{\sigma_R}{\omega_o}$, are plotted versus r, the ratio of the collision frequency of the gas to a characteristic frequency of the wave.

The dispersion coefficient predicted by the discrete ordinate solution is in very close agreement with the transformation theory for $r > 0.3$. The solution also agrees with the solution based on the Burnett equations for $r > 2$, and with the 13 moment solution for $r > 4$.

There is some spread in the absorption predicted by the various theories. The discrete ordinate solution is in qualitative agreement with the 13 moment and Burnett theories.

Since the absorption and dispersion predicted by the transformation theory for small values of r are obtained from the analytic continuation of the dispersion relation, it is felt that the transformation theory should be regarded as the standard against which other theories are compared. On this basis it may be concluded that the discrete ordinate solution yields results superior to the Burnett and 13 moment theories.

No experimental data could be found with which to compare the theories.

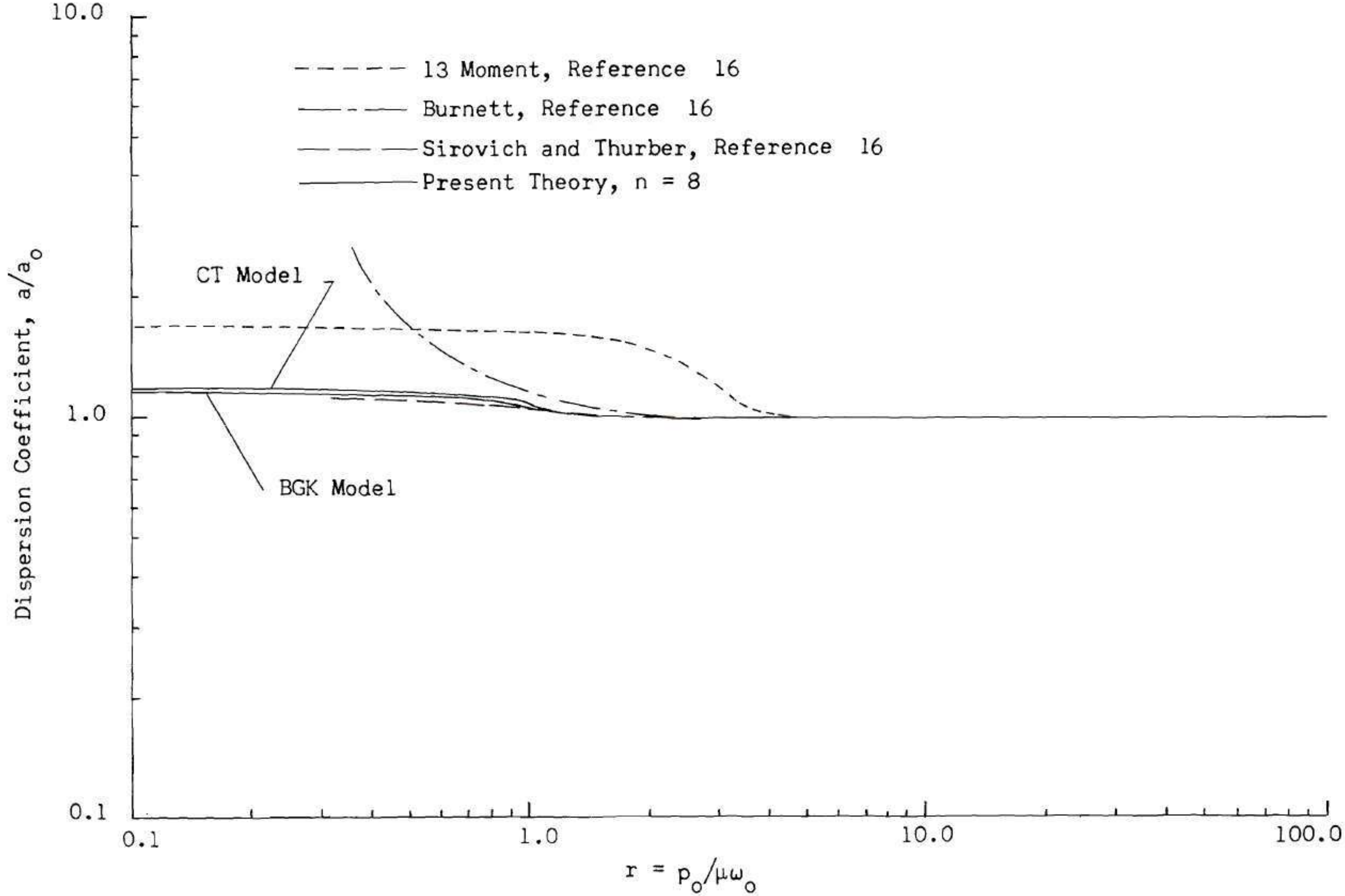

Figure 13. Comparison of Discrete Ordinate n = 8 Solution for Free Sound Wave Dispersion Coefficient With Theory.

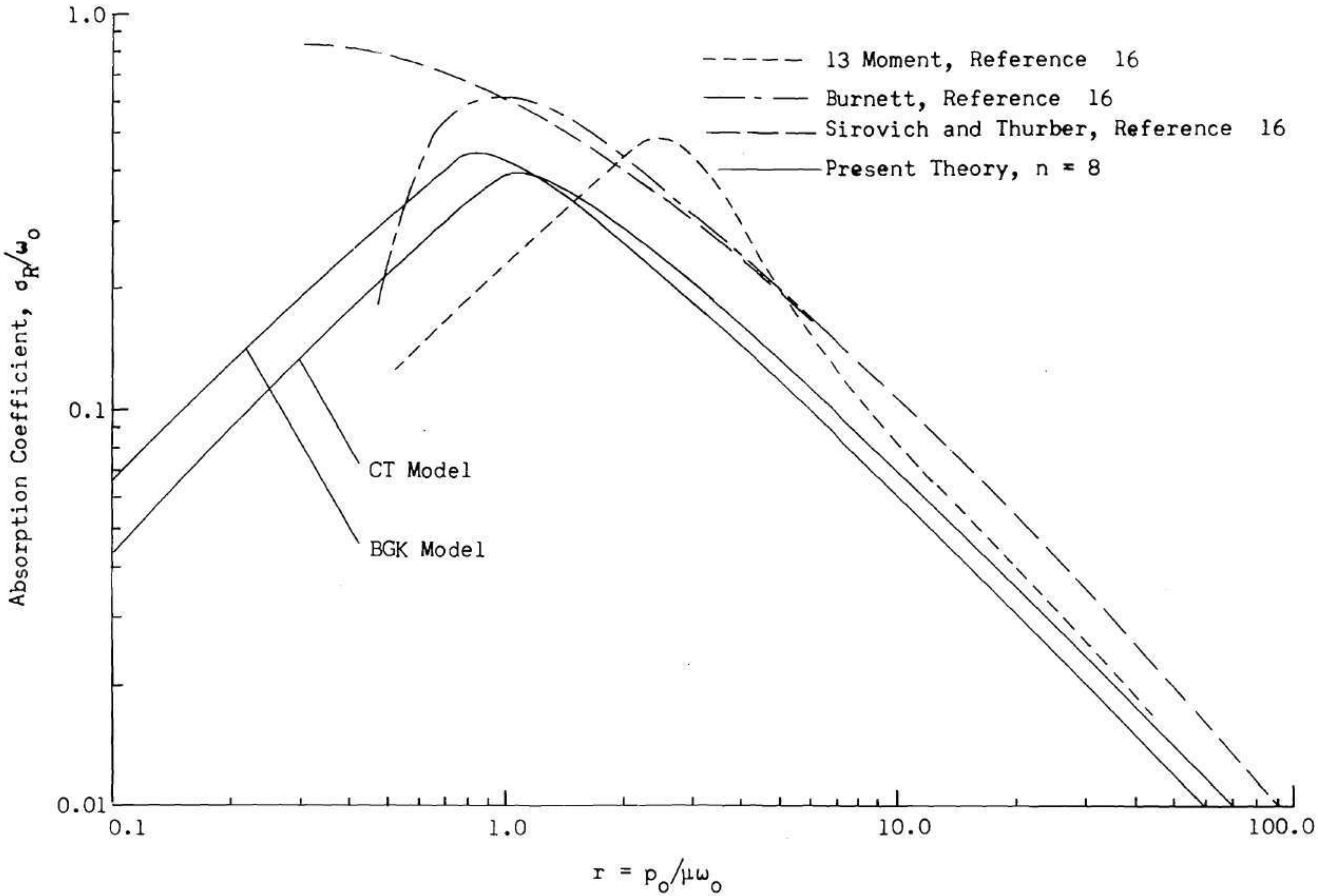

Figure 14. Comparison of Discrete Ordinate n = 8 Solution for Free Sound Wave Absorption Coefficient With Theory.

CHAPTER IV

FORCED SOUND WAVE PROPAGATION IN A MIXTURE OF GASES

Theoretical Formulation

In this chapter the problem of forced sound wave propagation in a mixture of monatomic gases will be considered. The mixture is composed of molecules of mass m and m^*. In the equilibrium state the temperature is T_o, and the partial pressures and number densities of the two components are given by (p_o, p_o^*) and (n_o, n_o^*), respectively. A small amplitude oscillation at an angular frequency, ω, perturbs the gas from equilibrium.

Discussion of the Model Equation

The analysis of this chapter is based upon the model equations which were proposed by Hamel [34]. These equations are of the same form as those postulated by Gross and Jackson [41], however, the determination of the collisional parameters is made on a more tenable basis. The model equation contains twenty-four collisional parameters. By requiring that the kinetic model provide for the conservation of the mass, momentum and energy of the total mixture, and separately the mass of each species, and satisfy the same symmetry property as the full collision integral, one obtains seventeen equations. The additional equations required are obtained by assuming the intermolecular force law is inverse fifth power, i.e., Maxwellian.

In the absence of body forces the governing equations may be written [34]#

$$\beta_o \frac{\partial\Phi}{\partial t} + c_x \frac{\partial\Phi}{\partial x} = \beta_o\left[n_o^*\chi^* + n_o\chi\right]\left[- \Phi + n' + 2c_x q_x + (c^2 - \frac{3}{2})T'\right] + \hat{m}^*\beta_o n_o^*\chi^*\left[- 2c_x q_x + 2(\frac{m}{m^*})^{1/2} c_x q_x^* + 2\hat{m}(c^2 - \frac{3}{2})(T^{*'} - T')\right] \tag{1}$$

and

$$\beta_o \frac{\partial\Phi^*}{\partial t} + (\frac{m}{m^*})^{1/2} c_x^* \frac{\partial\Phi^*}{\partial x} = \beta_o\left[n_o\chi^* + n_o^*\chi^{**}\right]\left[- \Phi^* + n^{*'} + 2c_x^* q_x^* + (c^{*2} - \frac{3}{2})T^{*'}\right] + \hat{m}\beta_o n_o\chi^* \left[- 2c_x^* q_x^* + 2(\frac{m^*}{m})^{1/2} c_x^* q_x + 2\hat{m}^*(c^{*2} - \frac{3}{2})(T' - T^{*'})\right] . \tag{2}$$

The functions Φ and Φ^* are perturbations to absolute Maxwellian distributions for the m and m^* molecules, respectively, and are defined by

$$f = \beta_o^3 f_o\left[1 + \Phi(x, \bar{c}, t)\right] \tag{3}$$

and

$$f^* = \beta_o^{*3} f_o^*\left[1 + \Phi^*(x, \bar{c}^*, t)\right] \tag{4}$$

It should be noted that Equation (16) of Reference [34] contains the incorrect term

$$\hat{m}^*\beta_o n_o^*\chi^*\left[2c_x q_x \hat{m} + 2(\frac{m}{m^*})^{1/2} \hat{m}^* c_x q_x^*\right] ;$$

the correct term is

$$\hat{m}^*\beta_o n_o^*\chi^*\left[- 2c_x q_x + 2(\frac{m}{m^*})^{1/2} c_x q_x^*\right] .$$

where

$$\beta_o = \left(\frac{m}{2kT_o}\right)^{1/2} , \tag{5}$$

$$\beta_o^* = \left(\frac{m^*}{2kT_o}\right)^{1/2} , \tag{6}$$

$$f_o = n_o \pi^{-3/2} e^{-c^2} \tag{7}$$

and

$$f_o^* = n_o^* \pi^{-3/2} e^{-c^{*2}} . \tag{8}$$

f and f^* are the distribution functions of the m and m^* molecules, respectively. The velocities $\bar{c}$ and $\bar{c}^*$ are the microscopic velocities of the m and m^* molecules nondimensionalized by $1/\beta_o$ and $1/\beta_o^*$, respectively. Perturbations to equilibrium values of the number density, macroscopic velocity and temperature are denoted by n', $n^{*\prime}$, q_x', $q_x^{*\prime}$, T' and $T^{*\prime}$ and are given by

$$n' = \pi^{-3/2} \int e^{-c_1^2} \Phi(x, \bar{c}_1, t) d^3 c_1 , \tag{9}$$

$$n^{*\prime} = \pi^{-3/2} \int e^{-c_1^{*2}} \Phi^*(x, \bar{c}_1^*, t) d^3 c_1^* , \tag{10}$$

$$q_x' = \pi^{-3/2} \int e^{-c_1^2} c_{x_1} \Phi(x, \bar{c}_1, t) d^3 c_1 , \tag{11}$$

$$q_x^{*'} = \pi^{-3/2} \int e^{-c_1^{*2}} c_{x_1}^* \Phi^*(x, \bar{c}_1^*, t) d^3 c_1^*, \tag{12}$$

$$T' = \frac{2}{3}\pi^{-3/2} \int e^{-c_1^2} (c_1^2 - \tfrac{3}{2})\Phi(x, \bar{c}_1, t) d^3 c_1 \tag{13}$$

and

$$T^{*'} = \frac{2}{3}\pi^{-3/2} \int e^{-c_1^{*2}} (c_1^{*2} - \tfrac{3}{2})\Phi^*(x, \bar{c}_1^*, t) d^3 c_1^* \quad . \tag{14}$$

respectively, where the shorthand notation

$$\int d^3 c_1 = \int_{-\infty}^{+\infty} dc_{x_1} \int_{-\infty}^{+\infty} dc_{y_1} \int_{-\infty}^{+\infty} dc_{z_1}$$

is used.

The parameters χ, χ^* and χ^{**} are related to the force constants for m-m, $m - m^*$ and $m^* - m^*$ collisions, respectively. A further discussion of these parameters is deferred until later.

The terms $\hat{m}$ and $\hat{m}^*$ are normalized reduced masses and are defined by

$$\hat{m} = \frac{m}{m + m^*} \tag{15}$$

and

$$\hat{m}^* = \frac{m^*}{m + m^*} \quad . \tag{16}$$

Normal Mode Assumption

A space and time dependence for Φ and Φ^* appropriate for forced wave motion is now introduced. It is assumed that

$$\Phi(x, \bar{c}, t) = \varphi(\bar{c})e^{\hat{i}(\omega t - Kx)} \tag{17}$$

and

$$\Phi^*(x, \bar{c}^*, t) = \varphi^*(\bar{c}^*)e^{\hat{i}(\omega t - Kx)} \quad , \tag{18}$$

where the functions φ and φ^* depend only on the molecular velocities of the m and m^* molecules, respectively, ω is the angular frequency and $\hat{i} = \sqrt{-1}$. K is the complex wave number which is to be determined.

By substituting Equations (17) and (18) into Equations (1) and (2), introducing Equations (9) through (14) and canceling the common exponential terms, one obtains

$$\left[\hat{i}\omega\beta_o - \hat{i}Kc_x + \beta_o\left(n_o^*\chi^* + n_o\chi\right)\right]\varphi(\bar{c}) = \tag{19}$$

$$\beta_o\left(n_o^*\chi^* + n_o\chi\right)\left[\pi^{-3/2}\int e^{-c_1^2}\varphi(\bar{c}_1)d^3c_1\right.$$

$$+ 2c_x\pi^{-3/2}\int e^{-c_1^2} c_{x_1}\varphi(\bar{c}_1)d^3c_1 + \tfrac{2}{3}(c^2 - \tfrac{3}{2})\pi^{-3/2}\int e^{-c_1^2}$$

$$\left.\cdot\,(c_1^2 - \tfrac{3}{2})\varphi(\bar{c}_1)d^3c_1\right] + \hat{m}^*\beta_o n_o^*\chi^*\left[-2c_x\pi^{-3/2}\int e^{-c_1^2} c_{x_1}\varphi(\bar{c}_1)d^3c_1\right.$$

$$+ 2\left(\frac{m}{m^*}\right)^{1/2} c_x\pi^{-3/2}\int e^{-c_1^{*2}} c_{x_1}^*\varphi^*(\bar{c}_1^*)d^3c_1^*$$

$$+ \frac{4}{3}\,\hat{m}(c^2 - \tfrac{3}{2})\Big(\pi^{-3/2}\int e^{-c_1^{*2}}(c_1^{*2} - \tfrac{3}{2})\varphi^*(\bar{c}_1^*)d^3c_1^*$$

$$-\pi^{-3/2}\int e^{-c_1^2}(c_1^2 - \tfrac{3}{2})\varphi(\bar{c}_1)d^3c_1\Big)\Big]$$

and

$$\Big[\hat{i}\omega\beta_o - \hat{i}K\left(\frac{m}{m^*}\right)^{1/2} c_x^* + \beta_o\Big(n_o\chi^* + n_o^*\chi^{**}\Big)\Big]\ \varphi^*(\bar{c}^*) = \tag{20}$$

$$\beta_o\Big(n_o\chi^* + n_o^*\chi^{**}\Big)\Big[\pi^{-3/2}\int e^{-c_1^{*2}}\varphi^*(\bar{c}_1^*)d^3c_1^*$$

$$+ 2c_x^*\pi^{-3/2}\int e^{-c_1^{*2}} c_{x_1}^*\varphi^*(\bar{c}_1^*)d^3c_1^* + \frac{2}{3}(c^{*2} - \tfrac{3}{2})\pi^{-3/2}\int e^{-c_1^{*2}}$$

$$\cdot\,(c_1^{*2} - \tfrac{3}{2})\varphi^*(\bar{c}_1^*)d^3c_1^*\Big] + \hat{m}\beta_o n_o\chi^*\Big[-2c_x^*\pi^{-3/2}\int e^{-c_1^{*2}} c_{x_1}^*\varphi^*(\bar{c}_1^*)d^3c_1^*$$

$$+ 2\left(\frac{m^*}{m}\right)^{1/2} c_x^*\pi^{-3/2}\int e^{-c_1^2} c_{x_1}\varphi(\bar{c}_1)d^3c_1$$

$$+ \frac{4}{3}\,\hat{m}^*(c^{*2} - \tfrac{3}{2})\Big(\pi^{-3/2}\int e^{-c_1^2}(c_1^2 - \tfrac{3}{2})\varphi(\bar{c}_1)d^3c_1$$

$$-\pi^{-3/2}\int e^{-c_1^{*2}}(c_1^{*2} - \tfrac{3}{2})\varphi^*(\bar{c}_1^*)d^3c_1^*\Big)\Big]$$

Transformation of the Integral Equations

As mentioned in previous chapters, it is possible to eliminate the velocity integrations perpendicular to the direction of propagation of the wave, and thereby reduce the magnitude of the system which results when the discrete ordinate approximation is used. The transformation used here is similar to that used earlier except that now one must introduce the four functions

$$\psi(c_x) = \pi^{-3/2} \int_{-\infty}^{+\infty} dc_y \int_{-\infty}^{+\infty} dc_z e^{-c_y^2 - c_z^2} \varphi(\bar{c}) \tag{21}$$

$$\eta(c_x) = \pi^{-3/2} \int_{-\infty}^{+\infty} dc_y \int_{-\infty}^{+\infty} dc_z e^{-c_y^2 - c_z^2} (c_y^2 + c_z^2) \varphi(\bar{c}) \tag{22}$$

$$\psi^*(c_x^*) = \pi^{-3/2} \int_{-\infty}^{+\infty} dc_y^* \int_{-\infty}^{+\infty} dc_z^* e^{-c_y^{*2} - c_z^{*2}} \varphi^*(\bar{c}^*) \tag{23}$$

and

$$\eta^*(c_x^*) = \pi^{-3/2} \int_{-\infty}^{+\infty} dc_y^* \int_{-\infty}^{+\infty} dc_z^* e^{-c_y^{*2} - c_z^{*2}} (c_y^{*2} + c_z^{*2}) \varphi^*(\bar{c}^*) \tag{24}$$

By integrating Equation (19) over (c_y, c_z) with respect to the two weighting functions $e^{-c_y^2 - c_z^2}$ and $(c_y^2 + c_y^2)e^{-c_y^2 - c_z^2}$ one obtains

$$\sqrt{\pi}\left[\hat{i}\omega\beta_o - \hat{i}Kc_x + \beta_o\left(n_o^*\chi^* + n_o\chi\right)\right] \psi(c_x) = \tag{25}$$

$$\beta_o\left(n_o^*\chi^* + n_o\chi\right)\left[\int_{-\infty}^{+\infty} e^{-c_{x_1}^2} \psi(c_{x_1}) dc_{x_1} + 2c_x \int_{-\infty}^{+\infty} e^{-c_{x_1}^2} c_{x_1} \psi(c_{x_1}) dc_{x_1}\right.$$

$$\left. + \frac{2}{3}(c_x^2 - \frac{1}{2}) \int_{-\infty}^{+\infty} e^{-c_{x_1}^2} \left[\eta(c_{x_1}) + (c_{x_1}^2 - \frac{3}{2})\psi(c_{x_1})\right] dc_{x_1}\right]$$

$$+ m^{*}\beta_o n_o^{*}\chi^{*}\Big[- 2c_x \int_{-\infty}^{+\infty} e^{-c_{x_1}^2} c_{x_1}\psi(c_{x_1})dc_{x_1}$$

$$+ 2\left(\frac{m}{m^*}\right)^{1/2} c_x \int_{-\infty}^{+\infty} e^{-c_{x_1}^{*2}} c_{x_1}^{*}\psi^{*}(c_{x_1}^{*})dc_{x_1}^{*}$$

$$+ \frac{4}{3}\hat{m}(c_x^2 - \tfrac{1}{2})\Big(\int_{-\infty}^{+\infty} e^{-c_{x_1}^{*2}} \Big[\eta^{*}(c_{x_1}^{*}) + (c_{x_1}^{*2} - \tfrac{3}{2})\psi^{*}(c_{x_1}^{*})\Big]dc_{x_1}^{*}$$

$$- \int_{-\infty}^{+\infty} e^{-c_{x_1}^2} \Big[\eta(c_{x_1}) + (c_{x_1}^2 - \tfrac{3}{2})\psi(c_{x_1})\Big]dc_{x_1}\Big)\Big],$$

and

$$\sqrt{\pi}\,\Big[\hat{i}\omega\beta_o - \hat{i}Kc_x + \beta_o\big(n_o^{*}\chi^{*} + n_o\chi\big)\Big]\,\eta(c_x) = \tag{26}$$

$$\beta_o\big(n_o^{*}\chi^{*} + n_o\chi\big)\Big[\int_{-\infty}^{+\infty} e^{-c_{x_1}^2} \psi(c_{x_1})dc_{x_1} + 2c_x \int_{-\infty}^{+\infty} e^{-c_{x_1}^2}$$

$$\cdot\, c_{x_1}\psi(c_{x_1})dc_{x_1} + \tfrac{2}{3}(c_x^2 + \tfrac{1}{2}) \int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\Big[\eta(c_{x_1})$$

$$+ (c_{x_1}^2 - \tfrac{3}{2})\psi(c_{x_1})\Big]dc_{x_1}\Big] + \hat{m}^{*}\beta_o n_o^{*}\chi^{*}\Big[- 2c_x \int_{-\infty}^{+\infty} e^{-c_{x_1}^2} c_{x_1}\psi(c_{x_1})dc_{x_1}$$

$$+ 2\left(\frac{m}{m^*}\right)^{1/2} c_x \int_{-\infty}^{+\infty} e^{-c_{x_1}^{*2}} c_{x_1}^{*}\psi^{*}(c_{x_1}^{*})dc_{x_1}^{*}$$

$$+ \frac{4}{3}\hat{m}(c_x^2 + \tfrac{1}{2})\Big(\int_{-\infty}^{+\infty} e^{-c_{x_1}^{*2}}\Big[\eta^{*}(c_{x_1}^{*}) + (c_{x_1}^{*2} - \tfrac{3}{2})\psi^{*}(c_{x_1}^{*})\Big]dc_{x_1}^{*}$$

$$-\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\left[\eta(c_{x_1}) + (c_{x_1}^2 - \frac{3}{2})\psi(c_{x_1})\right] dc_{x_1}\Big)\Big] .$$

To transform Equation (20), integrate over (c_y^*, c_z^*) with respect to the two weighting functions $e^{-c_y^{*2}-c_z^{*2}}$ and $(c_y^{*2} + c_z^{*2})e^{-c_y^{*2}-c_z^{*2}}$. Performing these integrations yields

$$\sqrt{\pi}\left[\hat{\imath}\omega\beta_o - \hat{\imath}K(\frac{m}{m^*})^{1/2} c_x^* + \beta_o\left(n_o\chi^* + n_o^*\chi^{**}\right)\right]\psi^*(c_x^*) = \tag{27}$$

$$\beta_o\left(n_o\chi^* + n_o^*\chi^{**}\right)\Big[\Big[\int_{-\infty}^{+\infty} e^{-c_{x_1}^{*2}}\psi^*(c_{x_1}^*)dc_{x_1}^* + 2c_x^*\int_{-\infty}^{+\infty} e^{-c_{x_1}^{*2}} c_{x_1}^*$$

$$\cdot\ \psi^*(c_{x_1}^*)dc_{x_1}^* + \frac{2}{3}(c_x^{*2} - \frac{1}{2})\int_{-\infty}^{+\infty} e^{-c_{x_1}^{*2}}\Big[\eta^*(c_{x_1}^*)$$

$$+ (c_{x_1}^{*2} - \frac{3}{2})\psi^*(c_{x_1}^*)\Big]dc_{x_1}^*\Big] + \hat{m}\beta_o n_o\chi^*\Big[-2c_x^*\int_{-\infty}^{+\infty} e^{-c_{x_1}^{*2}} c_{x_1}^*\psi^*(c_{x_1}^*)dc_{x_1}^*$$

$$+ 2(\frac{m^*}{m})^{1/2} c_x^*\int_{-\infty}^{+\infty} e^{-c_{x_1}^2} c_{x_1}\psi(c_{x_1})dc_{x_1}$$

$$+ \frac{4}{3}\hat{m}^*(c_x^{*2} - \frac{1}{2})\Big(\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\Big[\eta(c_{x_1}) + (c_{x_1}^2 - \frac{3}{2})\psi(c_{x_1})\Big]dc_{x_1}$$

$$- \int_{-\infty}^{+\infty} e^{-c_{x_1}^{*2}}\Big[\eta^*(c_{x_1}^*) + (c_{x_1}^{*2} - \frac{3}{2})\psi^*(c_{x_1}^*)\Big]dc_{x_1}^*\Big)\Big]$$

and

$$\sqrt{\pi}\left[\hat{i}\omega\beta_o - \hat{i}K\left(\frac{m}{m^*}\right)^{1/2} c_x^* + \beta_o\left(n_o\chi^* + n_o^*\chi^{**}\right)\right]\eta^*(c_x^*) = \tag{28}$$

$$+ \beta_o\left(n_o\chi^* + n_o^*\chi^{**}\right)\Big[\int_{-\infty}^{+\infty} e^{-c_{x_1}^{*2}} \psi^*(c_{x_1}^*)dc_{x_1}^*$$

$$+ 2c_x^* \int_{-\infty}^{+\infty} e^{-c_{x_1}^{*2}} c_{x_1}^* \psi^*(c_{x_1}^*)dc_{x_1}^*$$

$$+ \frac{2}{3}(c_x^{*2} + \frac{1}{2}) \int_{-\infty}^{+\infty} e^{-c_{x_1}^{*2}} \left[\eta^*(c_{x_1}^*) + (c_{x_1}^{*2} - \frac{3}{2})\psi^*(c_{x_1}^*)\right]dc_{x_1}^*\Big]$$

$$+ \hat{m}\beta_o n_o\chi^*\Big[- 2c_x^* \int_{-\infty}^{+\infty} e^{-c_{x_1}^{*2}} c_{x_1}^*\psi^*(c_{x_1}^*)dc_{x_1}^*$$

$$+ 2\left(\frac{m^*}{m}\right)^{1/2} c_x^* \int_{-\infty}^{+\infty} e^{-c_{x_1}^{2}} c_{x_1}\psi(c_{x_1})dc_{x_1}$$

$$+ \frac{4}{3}\,\hat{m}^*(c_x^{*2} + \frac{1}{2})\Big(\int_{-\infty}^{+\infty} e^{-c_{x_1}^{2}} \left[\eta(c_{x_1}) + (c_{x_1}^{2} - \frac{3}{2})\psi(c_{x_1})\right]dc_{x_1}$$

$$- \int_{-\infty}^{+\infty} e^{-c_{x_1}^{*2}}\left[\eta^*(c_{x_1}^*) + (c_{x_1}^{*2} - \frac{3}{2})\psi^*(c_{x_1}^*)\right]dc_{x_1}^*\Big)\Big].$$

Forced Sound Wave Parameters for a Mixture

As mentioned earlier, the parameters χ, χ^* and χ^{**} are related to the force constants for m - m, m - m^* and m^* - m^* collisions. The exact relations are [34]

$$\chi = 2.906(\zeta m)^{1/2}, \tag{29}$$

$$\chi^{**} = 2.906(\zeta^{**}m^*)^{1/2} \tag{30}$$

and

$$\chi^* = 2.66(\zeta^* m_o)^{1/2}, \tag{31}$$

where $m_o = m + m^*$. The terms ζ, ζ^{**} and ζ^* are defined by

$$F = \frac{m^2\zeta}{R^5}, \tag{32}$$

$$F^{**} = \frac{m^{*2}\zeta^{**}}{R^5} \tag{33}$$

and

$$F^* = \frac{mm^*\zeta^*}{R^5}, \tag{34}$$

where F, F^{**} and F^* are the forces at a distance R for m - m, $m^* - m^*$ and $m - m^*$ collisions, respectively. The self-collision constants, ζ and ζ^{**}, are readily obtained from the relations [44]

$$\mu = \frac{1}{3\pi}\left(\frac{2}{m\zeta}\right)^{1/2}\frac{kT_o}{A_2(5)} \tag{35}$$

and

$$\mu^* = \frac{1}{3\pi}\left(\frac{2}{m^*\zeta^{**}}\right)^{1/2}\frac{kT_o}{A_2(5)}, \tag{36}$$

where μ and μ^* are the viscosities of the m and m^* gases, respectively, and $A_2(5)$ is a pure number ($A_2(5) = 0.436$). k is Boltzmann's constant, and T_o is the equilibrium temperature.

In principle, it is possible to calculate the force constant for unlike molecules from the viscosity of the mixture. However, this calculation requires extremely accurate data for both pure components and their mixtures. For this reason, it is customary to make use of empirical "combining laws" which relate the force constant between unlike molecules to those between like molecules (see, e.g., Reference [45], page 567, or Reference [46], page 1071). If both pure components are assumed to have the same force law (Maxwellian, here), the constant ζ^* is given by

$$\zeta^* = (\zeta\zeta^{**})^{1/2} \quad . \tag{37}$$

By using Equations (29), (30), (35), and (36), one may show

$$n_o\chi = \frac{p_o}{\mu} \tag{38}$$

and

$$n_o^*\chi^{**} = \frac{p_o^*}{\mu^*} \quad , \tag{39}$$

where p_o and p_o^* are the partial pressures of the m and m^* molecules, respectively. It is now possible to introduce parameters r_M and r_M^* which are defined by

$$r_M = \frac{p_o}{\mu\omega} \tag{40}$$

and

$$r_M^* = \frac{p_o^*}{\mu^* \omega} \quad . \tag{41}$$

Note the similarity between these parameters and the previously used parameter r. In terms of r_M and r_M^*, $n_o\chi$ and $n_o^*\chi^{**}$ may be written

$$n_o\chi = \omega r_M \tag{42}$$

$$n_o^*\chi^{**} = \omega r_M^* \tag{43}$$

The cross-collision parameters, $n_o\chi^*$ and $n^*_o\chi^*$, may be expressed as

$$n_o\chi^* = \omega r_M(\frac{\chi^*}{\chi}) \tag{44}$$

and

$$n_o^*\chi^* = \omega r_M^*(\frac{\chi^*}{\chi^{**}}) \ , \tag{45}$$

where

$$\frac{\chi^*}{\chi} = \frac{2.66}{2.906} (\frac{m + m^*}{m})^{1/2} (\frac{m}{m^*})^{1/4} (\frac{\mu}{\mu^*})^{1/2} \tag{46}$$

and

$$\frac{\chi^*}{\chi^{**}} = \frac{2.66}{2.906}(\frac{m + m^*}{m^*})^{1/2} (\frac{m^*}{m})^{1/4} (\frac{\mu^*}{\mu})^{1/2} \quad . \tag{47}$$

Equations (46) and (47) are readily obtained from Equations (29), (30), (31), (35), (36) and (37).

In a given physical situation, the percentage of each component is given. One way to do this is by specifying the number density fractions of the components, α and α^*, which are defined by

$$\alpha = \frac{n_o}{n_o + n_o^*} \tag{48}$$

and

$$\alpha^* = \frac{n_o^*}{n_o + n_o^*} \quad . \tag{49}$$

Clearly $\alpha + \alpha^* = 1$, so that one need only consider α.

By using Equations (48) and (49), it is easily shown that the parameters r_M and r_M^* are related as follows:

$$r_M^* = \frac{1 - \alpha}{\alpha} \frac{\mu}{\mu^*} r_M \quad . \tag{50}$$

It is thus possible to express the four collision parameters $n_o\chi$, $n_o\chi^*$, $n_o^*\chi^*$ and $n_o^*\chi^{**}$ in terms of the two parameters r_M and α and the masses and viscosities of the two gases.

A nondimensional wave number, $\hat{K}_M$, is defined by

$$\hat{K}_M = \frac{KA_o}{\omega} \quad , \tag{51}$$

where

$$A_o = \sqrt{\frac{5/3\ kT_o}{\alpha m + \alpha^* m^*}} \quad . \tag{52}$$

Note that A_o is the adiabatic speed of sound for a mixture in the continuum limit.

<u>Discrete Ordinate Solution</u>

Equations (25) through (28) are now solved by the method of discrete ordinates (see Chapter II for a discussion of the method). By applying the discrete ordinate method and introducing Equations (44), (45) and (51), one obtains the four equations:

$$\left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2} \frac{1 - \hat{i}\dfrac{\chi^*}{\chi^{**}} r_M^* - \hat{i} r_M}{c_\ell} \psi_\ell \tag{53}$$

$$+ \hat{i}\left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2} \frac{1}{\sqrt{\pi}\ c_\ell} \sum_{i=1}^{2n} H_i \Bigg((r_M^* \frac{\chi^*}{\chi^{**}} + r_M)\Big[1 + 2c_\ell c_i$$

$$+ \frac{2}{3}(c_\ell^2 - \frac{1}{2})(c_i^2 - \frac{3}{2})\Big] - 2\hat{m}^* \frac{\chi^*}{\chi^{**}} r_M^* c_\ell c_i - \frac{4}{3}\hat{m}^*\hat{m}\frac{\chi^*}{\chi^{**}} r_M^*$$

$$\cdot (c_\ell^2 - \frac{1}{2})(c_i^2 - \frac{3}{2})\Bigg)\psi_i + \hat{i}\left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2} \frac{1}{\sqrt{\pi}\ c_\ell} \sum_{i=1}^{2n} H_i \Bigg($$

$$\frac{2}{3}(r_M^* \frac{\chi^*}{\chi^{**}} + r_M)(c_\ell^2 - \frac{1}{2}) - \frac{4}{3}\hat{m} r_M^* \frac{\chi^*}{\chi^{**}}(c_\ell^2 - \frac{1}{2})\Bigg)\eta_i$$

$$+ \hat{i}\left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2} \frac{1}{\sqrt{\pi}\ c_\ell} \sum_{i=1}^{2n} H_i \Bigg(2\hat{m}^* (\frac{m}{m^*})^{1/2} r_M^* \frac{\chi^*}{\chi^{**}} c_\ell c_i^*$$

$$+ \frac{4}{3}\,\hat{m}^*\hat{m}r_M^*\frac{\chi^*}{\chi^{**}}\,(c_\ell^2 - \tfrac{1}{2})(c_i^{*2} - \tfrac{3}{2})\Big)\psi_i^*$$

$$+ \hat{i}\left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2}\frac{1}{\sqrt{\pi}\;c_\ell}\sum_{i=1}^{2n} H_i\left(\frac{4}{3}\,\hat{m}^*\hat{m}r_M^*\frac{\chi^*}{\chi^{**}}\,(c_\ell^2 - \tfrac{1}{2})\right)\eta_i^* = \hat{K}_M\psi_\ell$$

$$\hat{i}\left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2}\frac{1}{\sqrt{\pi}\;c_\ell}\sum_{i=1}^{2n} H_i\Big((r_M^*\frac{\chi^*}{\chi^{**}} + r_M)\Big[1 + 2c_\ell c_i \tag{54}$$

$$+ \frac{2}{3}(c_\ell^2 + \tfrac{1}{2})(c_i^2 - \tfrac{3}{2})\Big] - 2\hat{m}^* r_M^*\frac{\chi^*}{\chi^{**}}\,c_\ell c_i - \frac{4}{3}\,\hat{m}^*\hat{m}$$

$$\cdot\; r_M^*\frac{\chi^*}{\chi^{**}}(c_\ell^2 + \tfrac{1}{2})(c_i^2 - \tfrac{3}{2})\Big)\psi_i + \left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2}$$

$$\cdot\;\frac{1 - \hat{i}r_M^*\frac{\chi^*}{\chi^{**}} - \hat{i}r_M}{c_\ell}\;\eta_\ell + \hat{i}\left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2}\frac{1}{\sqrt{\pi}\;c_\ell}$$

$$\cdot\;\sum_{i=1}^{2n} H_i\left(\frac{2}{3}(r_M^*\frac{\chi^*}{\chi^{**}} + r_M)(c_\ell^2 + \tfrac{1}{2}) - \frac{4}{3}\,\hat{m}^*\hat{m}r_M^*\frac{\chi^*}{\chi^{**}}(c_\ell^2 + \tfrac{1}{2})\right)\eta_i$$

$$+ \hat{i}\left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2}\frac{1}{\sqrt{\pi}\;c_\ell}\sum_{i=1}^{2n} H_i\Big(2\hat{m}^*\left(\frac{m}{m^*}\right)^{1/2} r_M^*\frac{\chi^*}{\chi^{**}}$$

$$\cdot\; c_\ell c_i^* + \frac{4}{3}\,\hat{m}^*\hat{m}r_M^*\frac{\chi^*}{\chi^{**}}\,(c_\ell^2 + \tfrac{1}{2})(c_i^{*2} - \tfrac{3}{2})\Big)\,\psi_i^* + \hat{i}\left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2}$$

$$\cdot\;\frac{1}{\sqrt{\pi}\;c_\ell}\sum_{i=1}^{2n} H_i\left(\frac{4}{3}\,\hat{m}^*\hat{m}r_M^*\frac{\chi^*}{\chi^{**}}(c_\ell^2 + \tfrac{1}{2})\right)\eta_i^* = \hat{K}_M\eta_\ell$$

$$\hat{i}\left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2} \frac{1}{\sqrt{\pi}\; c_\ell} \left(\frac{m^*}{m}\right)^{1/2} \sum_{i=1}^{2n} H_i\left(2\hat{m}\left(\frac{m^*}{m}\right)^{1/2} r_M \frac{\chi^*}{\chi}\, c_\ell^* c_i \right. \tag{55}$$

$$\left. + \frac{4}{3}\,\hat{m}^*\hat{m} r_M \frac{\chi^*}{\chi}\left(c_\ell^{*2} - \frac{1}{2}\right)\left(c_i^2 - \frac{3}{2}\right)\right)\psi_i + \hat{i}\left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2} \frac{1}{\sqrt{\pi}\; c_\ell^*}$$

$$\cdot \left(\frac{m^*}{m}\right)^{1/2} \sum_{i=1}^{2n} H_i\left(\frac{4}{3}\,\hat{m}^*\hat{m} r_M \frac{\chi^*}{\chi}\left(c_\ell^{*2} - \frac{1}{2}\right)\right)\eta_i$$

$$+ \left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2} \left(\frac{m^*}{m}\right)^{1/2} \frac{1 - \hat{i} r_M \frac{\chi^*}{\chi} - \hat{i} r_M^*}{c_\ell^*}\,\psi_\ell^*$$

$$+ \hat{i}\left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2} \frac{1}{\sqrt{\pi}\; c_\ell^*} \left(\frac{m^*}{m}\right)^{1/2} \sum_{i=1}^{2n} H_i\left(\left(r_M \frac{\chi^*}{\chi} + r_M^*\right)\right.$$

$$\cdot \left[1 + 2c_\ell^* c_i^* + \frac{2}{3}\left(c_\ell^{*2} - \frac{1}{2}\right)\left(c_i^{*2} - \frac{3}{2}\right)\right] - 2\hat{m} r_M \left(\frac{\chi^*}{\chi}\right) c_\ell^* c_i^*$$

$$\left. - \frac{4}{3}\,\hat{m}^*\hat{m} r_M \left(\frac{\chi^*}{\chi}\right)\left(c_\ell^{*2} - \frac{1}{2}\right)\left(c_i^{*2} - \frac{3}{2}\right)\right)\psi_i^* + \hat{i}\left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2}$$

$$\cdot \frac{1}{\sqrt{\pi}\; c_\ell^*} \left(\frac{m^*}{m}\right)^{1/2} \sum_{i=1}^{2n} H_i\left(\frac{2}{3}\left(r_M \frac{\chi^*}{\chi} + r_M^*\right)\left(c_\ell^{*2} - \frac{1}{2}\right) \right.$$

$$\left. - \frac{4}{3}\,\hat{m}^*\hat{m} r_M \frac{\chi^*}{\chi}\left(c_\ell^{*2} - \frac{1}{2}\right)\right)\eta_i^* = \hat{K}_M \psi_\ell^*$$

$$
\begin{aligned}
&\hat{i}\left[\frac{5m}{6(\alpha m + \alpha^{*} m^{*})}\right]^{1/2} \frac{1}{\sqrt{\pi}\; c_{\ell}^{*}} \left(\frac{m^{*}}{m}\right)^{1/2} \sum_{i=1}^{2n} H_i \Bigg(2\hat{m}\left(\frac{m^{*}}{m}\right)^{1/2} r_M \frac{\chi^{*}}{\chi} \\
&\cdot c_{\ell}^{*} c_i + \frac{4}{3}\hat{m}^{*}\hat{m} r_M \frac{\chi^{*}}{\chi}\left(c_{\ell}^{*2} + \frac{1}{2}\right)\left(c_i^{2} - \frac{3}{2}\right)\Bigg)\psi_i + \hat{i}\left[\frac{5m}{6(\alpha m + a^{*} m^{*})}\right]^{1/2} \\
&\cdot \frac{1}{\sqrt{\pi}\; c_{\ell}} \left(\frac{m^{*}}{m}\right)^{1/2} \sum_{i=1}^{2n} H_i \Bigg(\frac{4}{3}\hat{m}^{*}\hat{m} r_M \frac{\chi^{*}}{\chi}\left(c_{\ell}^{*2} + \frac{1}{2}\right)\Bigg)\eta_i \\
&+ \hat{i}\left[\frac{5m}{6(\alpha m + \alpha^{*} m^{*})}\right]^{1/2} \frac{1}{\sqrt{\pi}\; c_{\ell}^{*}} \left(\frac{m^{*}}{m}\right)^{1/2} \sum_{i=1}^{2n} H_i \Bigg(\left(r_M \frac{\chi^{*}}{\chi} + r_M^{*}\right) \\
&\cdot \left[1 + 2c_{\ell}^{*} c_i^{*} + \frac{2}{3}\left(c_{\ell}^{*2} + \frac{1}{2}\right)\left(c_i^{*2} - \frac{3}{2}\right)\right] - 2\hat{m} r_M \frac{\chi^{*}}{\chi} c_{\ell}^{*} c_i^{*} \\
&- \frac{4}{3}\hat{m}^{*}\hat{m} r_M \frac{\chi^{*}}{\chi}\left(c_{\ell}^{*2} + \frac{1}{2}\right)\left(c_i^{*2} - \frac{3}{2}\right)\Bigg)\psi_i^{*} + \left[\frac{5m}{6(\alpha m + \alpha^{*} m^{*})}\right]^{1/2} \left(\frac{m^{*}}{m}\right)^{1/2} \\
&\cdot \frac{1 - \hat{i} r_M \frac{\chi^{*}}{\chi} - \hat{i} r_M^{*}}{c_{\ell}^{*}} \eta_{\ell}^{*} + \hat{i}\left[\frac{5m}{6(\alpha m + \alpha^{*} m^{*})}\right]^{1/2} \frac{1}{\sqrt{\pi}\; c_{\ell}^{*}} \left(\frac{m^{*}}{m}\right)^{1/2} \\
&\cdot \sum_{i=1}^{2n} H_i \Bigg(\frac{2}{3}\left(r_M \frac{\chi^{*}}{\chi} + r_M^{*}\right)\left(c_{\ell}^{*2} + \frac{1}{2}\right) - \frac{4}{3}\hat{m}^{*}\hat{m} r_M \frac{\chi^{*}}{\chi} \\
&\cdot \left(c_{\ell}^{*2} + \frac{1}{2}\right)\Bigg)\eta_i^{*} = \hat{K}_M \eta_{\ell}^{*} \; .
\end{aligned}
\tag{56}
$$

All terms introduced by the discrete ordinate approximation were discussed in Chapter II, and therefore they are not discussed again here.

As was the case in Chapter II, the system of equations represented by Equations (53), (54), (55) and (56) will have a nontrivial solution only if $\hat{K}_M$ is an eigenvalue of the matrix $\bar{\bar{A}}$, where $\bar{\bar{A}}$ has the elements

$$A_{\ell,i} = \left[\frac{5m}{6(\alpha m + a^* m^*)}\right]^{1/2} \frac{1 - \hat{i}\frac{\chi^*}{\chi^{**}} r_M^* - \hat{i} r_M}{c_\ell} \delta_{\ell i} \tag{57}$$

$$+ \hat{i}\left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2} \frac{1}{\sqrt{\pi}\; c_\ell} H_i\Bigg((r_M^* \frac{\chi^*}{\chi^{**}} + r_M)$$

$$\cdot \left[1 + 2c_\ell c_i + \frac{2}{3}(c_\ell^2 - \frac{1}{2})(c_i^2 - \frac{3}{2})\right] - 2\hat{m}^* \frac{\chi^*}{\chi^{**}} r_M^* c_\ell c_i$$

$$- \frac{4}{3}\hat{m}^*\hat{m}\frac{\chi^*}{\chi^{**}} r_M^*(c_\ell^2 - \frac{1}{2})(c_i^2 - \frac{3}{2})\Bigg)$$

$$A_{\ell,i+2n} = \hat{i}\left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2} \frac{1}{\sqrt{\pi}\; c_\ell} H_i\Bigg(\frac{2}{3}(r_M^*\frac{\chi^*}{\chi^{**}} + r_M)(c_\ell^2 - \frac{1}{2})$$

$$- \frac{4}{3}\hat{m} r_M^* \frac{\chi^*}{\chi^{**}}(c_\ell^2 - \frac{1}{2})\Bigg) \tag{58}$$

$$A_{\ell,i+4n} = \hat{i}\left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2} \frac{1}{\sqrt{\pi}\; c_\ell} H_i\Bigg(2\hat{m}^*\left(\frac{m}{m^*}\right)^{1/2} r_M^* \frac{\chi^*}{\chi^{**}} \tag{59}$$

$$\cdot\; c_\ell c_i^* + \frac{4}{3}\hat{m}^*\hat{m} r_M^* \frac{\chi^*}{\chi^{**}}(c_\ell^2 - \frac{1}{2})(c_i^{*2} - \frac{3}{2})\Bigg)$$

$$A_{\ell,i+6n} = \hat{i}\left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2} \frac{1}{\sqrt{\pi}\; c_\ell} H_i\left(\frac{4}{3}\hat{m}^*\hat{m} r_M^* \frac{\chi^*}{\chi^{**}}(c_\ell^2 - \frac{1}{2})\right) \tag{60}$$

$$A_{\ell+2n,i} = \hat{i}\left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2} \frac{1}{\sqrt{\pi}\ c_\ell} \cdot H_i\Bigg((r_M^* \frac{\chi^*}{\chi^{**}} + r_M) \tag{61}$$

$$\cdot \left[1 + 2c_\ell c_i + \frac{2}{3}(c_\ell^2 + \frac{1}{2})(c_i^2 - \frac{3}{2})\right] - 2\hat{m}^* r_M^* \frac{\chi^*}{\chi^{**}} c_\ell c_i$$

$$- \frac{4}{3}\hat{m}^*\hat{m} r_M^* \frac{\chi^*}{\chi^{**}}(c_\ell^2 + \frac{1}{2})(c_i^2 - \frac{3}{2})\Bigg)$$

$$A_{\ell+2n,i+2n} = \left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2} \frac{1 - \hat{i} r_M^* \frac{\chi^*}{\chi^{**}} - \hat{i} r_M}{c_\ell} \delta_{\ell i} \tag{62}$$

$$+ \hat{i}\left[\frac{5m}{6(\alpha m + \alpha^* m)}\right]^{1/2} \frac{1}{\sqrt{\pi}\ c_\ell} \cdot H_i$$

$$\cdot \Bigg(\frac{2}{3}(r_M^* \frac{\chi^*}{\chi^{**}} + r_M)(c_\ell^2 + \frac{1}{2})$$

$$- \frac{4}{3}\hat{m}^*\hat{m} r_M^* \frac{\chi^*}{\chi^{**}}(c_\ell^2 + \frac{1}{2})\Bigg)$$

$$A_{\ell+2n,i+4n} = \hat{i}\left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2} \frac{1}{\sqrt{\pi}\ c_\ell} H_i\Bigg(2\hat{m}^* (\frac{m}{m^*})^{1/2} \tag{63}$$

$$\cdot r_M^* \frac{\chi^*}{\chi^{**}} c_\ell c_i^* + \frac{4}{3}\hat{m}^*\hat{m} r_M^* \frac{\chi^*}{\chi^{**}}(c_\ell^2 + \frac{1}{2})(c_i^{*2} - \frac{3}{2})\Bigg)$$

$$A_{\ell+2n,i+6n} = \hat{i}\left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2} \frac{1}{\sqrt{\pi}\ c_\ell} H_i\Bigg(\frac{4}{3}\hat{m}^*\hat{m} r_M^* \frac{\chi^*}{\chi^{**}} \tag{64}$$

$$\cdot (c_\ell^2 + \frac{1}{2})\Bigg)$$

$$A_{\ell+4n,i} = \hat{i}\left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2} \frac{1}{\sqrt{\pi}\; c_\ell}\left(\frac{m^*}{m}\right)^{1/2} H_i\left(2\hat{m}\left(\frac{m^*}{m}\right)^{1/2}\right. \tag{65}$$

$$\left. \cdot\; r_M \frac{\chi^*}{\chi}\, c_\ell^* c_i + \frac{4}{3}\hat{m}^*\hat{m} r_M \frac{\chi^*}{\chi}(c_\ell^{*2} - \tfrac{1}{2})(c_i^2 - \tfrac{3}{2})\right)$$

$$A_{\ell+4n,i+2n} = \hat{i}\left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2} \frac{1}{\sqrt{\pi}\; c_\ell^*}\left(\frac{m^*}{m}\right)^{1/2} \tag{66}$$

$$\cdot\; H_i \frac{4}{3}\hat{m}^*\hat{m} r_M \frac{\chi^*}{\chi}(c_\ell^{*2} - \tfrac{1}{2})$$

$$A_{\ell+4n,i+4n} = \left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2} \left(\frac{m^*}{m}\right)^{1/2} \frac{1 - \hat{i} r_M \frac{\chi^*}{\chi} - \hat{i} r_M^*}{c_\ell^*}\,\delta_{\ell i} \tag{67}$$

$$+ \hat{i}\left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2} \frac{1}{\sqrt{\pi}\; c_\ell^*}\left(\frac{m^*}{m}\right)^{1/2}$$

$$\cdot\; H_i\left((r_M \frac{\chi^*}{\chi} + r_M^*)\left[1 + 2c_\ell^* c_i^* + \frac{2}{3}(c_\ell^{*2} - \tfrac{1}{2})(c_i^{*2} - \tfrac{3}{2})\right]\right.$$

$$\left. -\; 2\hat{m} r_M\left(\frac{\chi^*}{\chi}\right) c_\ell^* c_i^* - \frac{4}{3}\hat{m}^*\hat{m} r_M\left(\frac{\chi^*}{\chi}\right)(c_\ell^{*2} - \tfrac{1}{2})(c_i^{*2} - \tfrac{3}{2})\right)$$

$$A_{\ell+4n,i+6n} = \hat{i}\left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2} \frac{1}{\sqrt{\pi}\; c_\ell^*}\left(\frac{m^*}{m}\right)^{1/2} \tag{68}$$

$$\cdot\; H_i\left(\frac{2}{3}(r_M \frac{\chi^*}{\chi} + r_M^*)(c_\ell^{*2} - \tfrac{1}{2}) - \frac{4}{3}\hat{m}^*\hat{m} r_M \frac{\chi^*}{\chi}(c_\ell^{*2} - \tfrac{1}{2})\right)$$

$$A_{\ell+6n,i} = \hat{i}\left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2} \frac{1}{\sqrt{\pi}\, c_\ell^*}\left(\frac{m^*}{m}\right)^{1/2} H_i\left(2\hat{m}\left(\frac{m^*}{m}\right)^{1/2}\right.$$

$$\left. \cdot\, r_M \frac{\chi^*}{\chi}\, c_\ell^* c_i + \frac{4}{3}\hat{m}^*\hat{m} r_M \frac{\chi^*}{\chi}(c_\ell^{*2} + \tfrac{1}{2})(c_i^2 - \tfrac{3}{2})\right) \tag{69}$$

$$A_{\ell+6n,i+2n} = \hat{i}\left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2} \frac{1}{\sqrt{\pi}\, c_\ell}\left(\frac{m^*}{m}\right)^{1/2} H_i\left(\frac{4}{3}\hat{m}^*\hat{m} r_M\right.$$

$$\left. \cdot\, \frac{\chi^*}{\chi}(c_\ell^{*2} + \tfrac{1}{2})\right) \tag{70}$$

$$A_{\ell+6n,i+4n} = \hat{i}\left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2} \frac{1}{\sqrt{\pi}\, c_\ell^*}\left(\frac{m^*}{m}\right)^{1/2} H_i\left((r_M \frac{\chi^*}{\chi}\right.$$

$$+ r_M^*)\left[1 + 2c_\ell^* c_i^* + \tfrac{2}{3}(c_\ell^{*2} + \tfrac{1}{2})(c_i^{*2} - \tfrac{3}{2})\right]$$

$$\left. - 2\hat{m} r_M \frac{\chi^*}{\chi}\, c_\ell^* c_i^* - \frac{4}{3}\hat{m}^*\hat{m} r_M \frac{\chi^*}{\chi}(c_\ell^{*2} + \tfrac{1}{2})(c_i^{*2} - \tfrac{3}{2})\right) \tag{71}$$

$$A_{\ell+6n,i+6n} = \left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2} \left(\frac{m^*}{m}\right)^{1/2} \frac{1 - \hat{i} r_M \frac{\chi^*}{\chi} - \hat{i} r_M}{c_\ell^*}\, \delta_{\ell i}$$

$$+ \hat{i}\left[\frac{5m}{6(\alpha m + \alpha^* m^*)}\right]^{1/2} \frac{1}{\sqrt{\pi}\, c_\ell^*}\left(\frac{m^*}{m}\right)^{1/2} H_i\left(\frac{2}{3}(r_M \frac{\chi^*}{\chi}\right.$$

$$\left. + r_M^*)(c_\ell^{*2} + \tfrac{1}{2}) - \frac{4}{3}\hat{m}^*\hat{m} r_M \frac{\chi^*}{\chi}(c_\ell^{*2} + \tfrac{1}{2})\right) \tag{72}$$

where

$\ell = (1, 2, \ldots, 2n)$, $i = (1, 2, \ldots, 2n)$

and n is the number of positive discrete velocity points.

Computational Procedure

In order to calculate the matrix $\bar{\bar{A}}$ it is necessary to specify the constituents of the gas mixture (m and m^*), their number density ratios (α and α^*), their viscositiies|| (μ and μ^*), the parameter r_M and the number of positive discrete velocity points, n. Once these specifications have been made, a straightforward variation of ℓ and i in Equations (57) through (72) yields all of the elements of $\bar{\bar{A}}$.

Discussion of Results

The absorption and dispersion coefficients were calculated for Argon-Helium, Neon-Helium and Argon-Neon mixtures. Mixture ratios of 0.25, 0.50 and 0.75 were considered (in all cases, α denotes the number density fraction of the lighter element). The parameter r_M was varied from 1000.0 to 1.0. All calculations are for n = 4.

Figures 15 and 16 show the results obtained for an Argon-Helium mixture. In these figures the dispersion coefficient, A_o/a, and the absorption coefficient, $-K_I A_o/\omega$, are plotted versus r_M. The experimental data are those of Holmes and Tempest [19]. As may be seen, the discrete ordinate solution is in excellent agreement with the experimental data.

Similar remarks hold for the Neon-Helium results, which are presented in Figures 17 and 18.

|| The viscosities were obtained from Reference [45] and are, of course, temperature dependent. However, Equations (57) through (72) depend on the ratio of the two viscosities, and this ratio is nearly independent of temperature in the temperature range 0°C < T < 150°C. Thus, although the calculations were based on the viscosity at T_o = 25°C (the temperature at which the experiment was conducted), the results are applicable over a wider range of temperature.

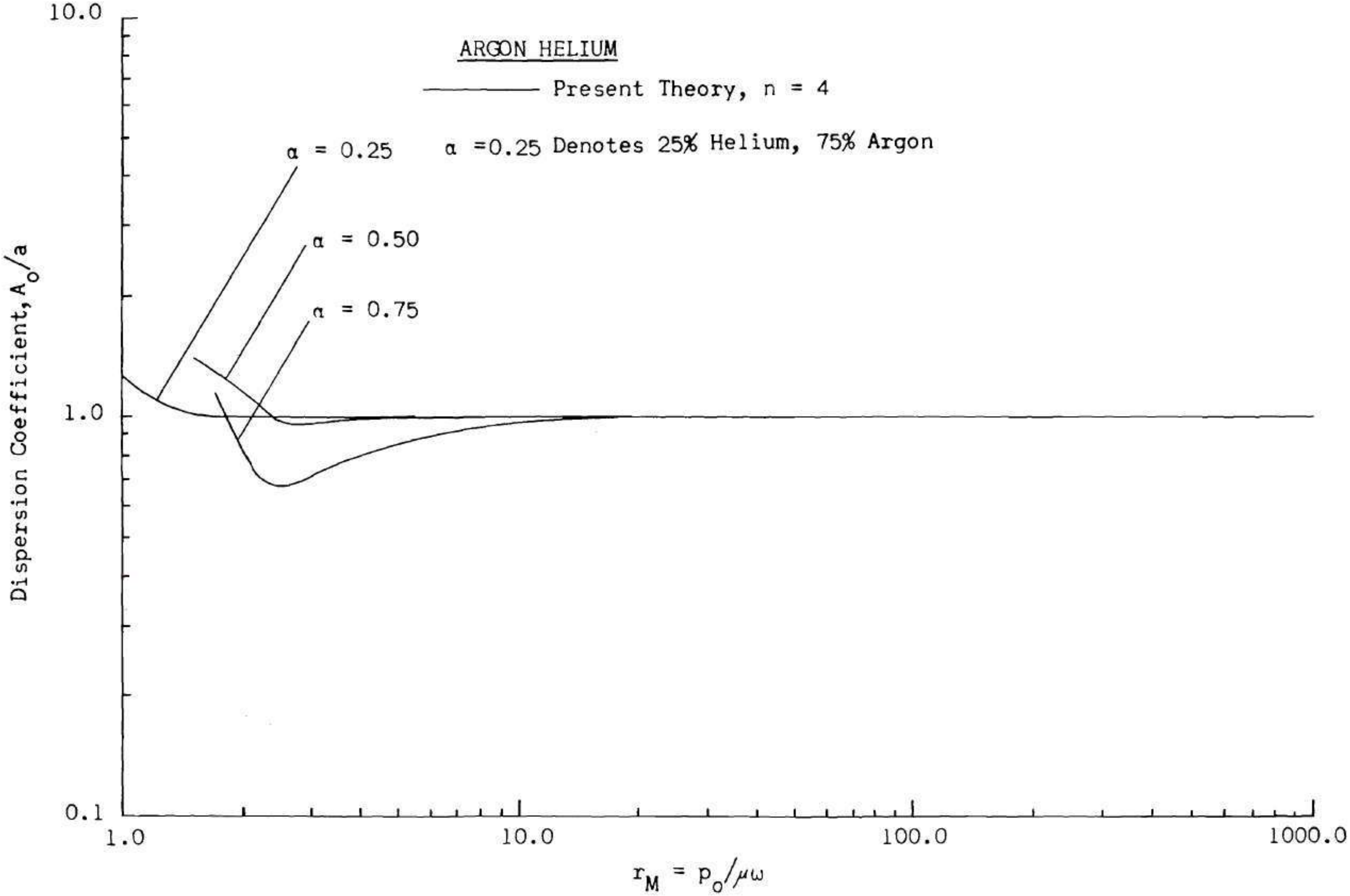

Figure 15. Comparison of the Discrete Ordinate Solution for the Dispersion Coefficient in an Argon-Helium Mixture with Experiment.

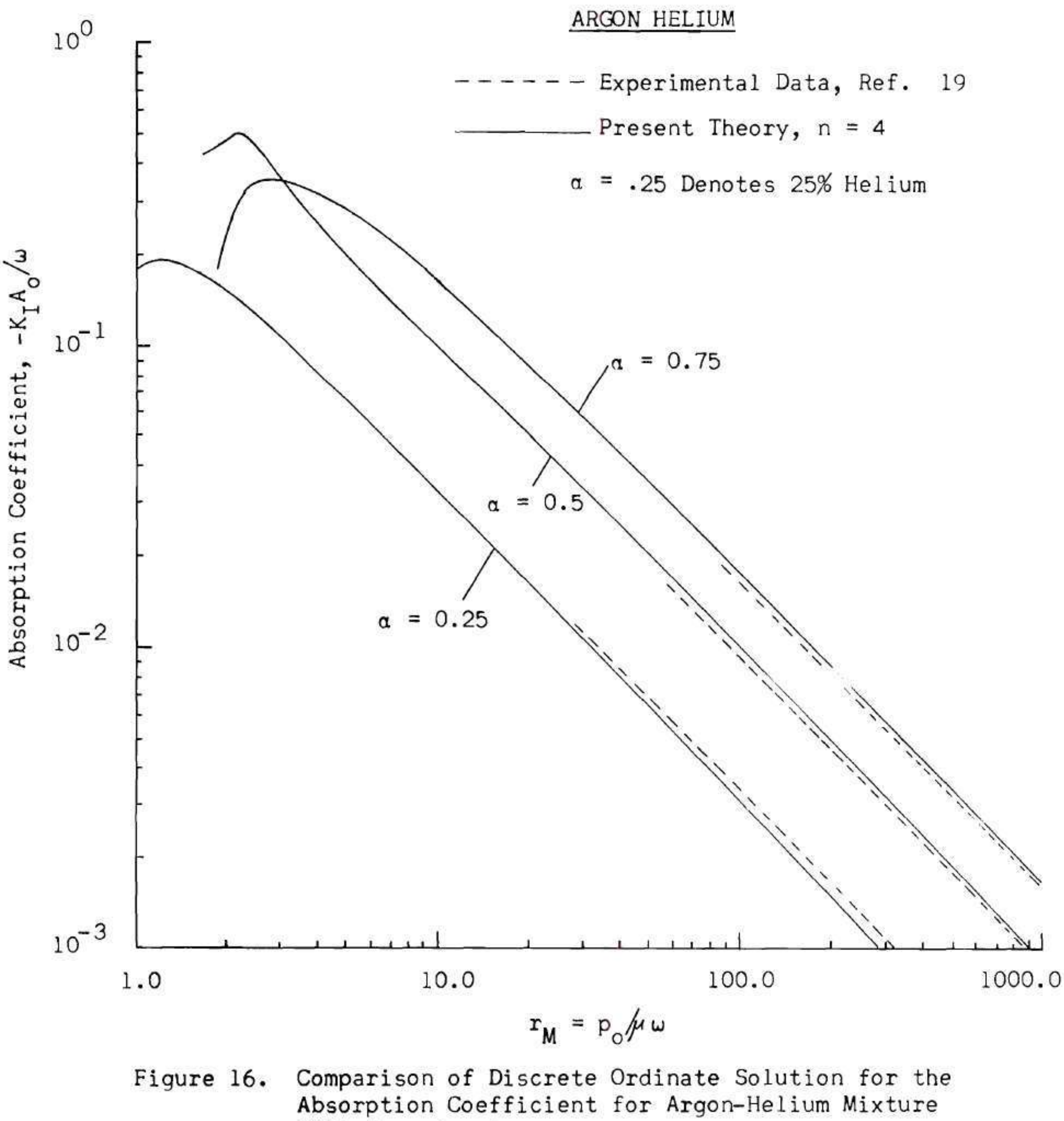

Figure 16. Comparison of Discrete Ordinate Solution for the Absorption Coefficient for Argon-Helium Mixture With Experiment.

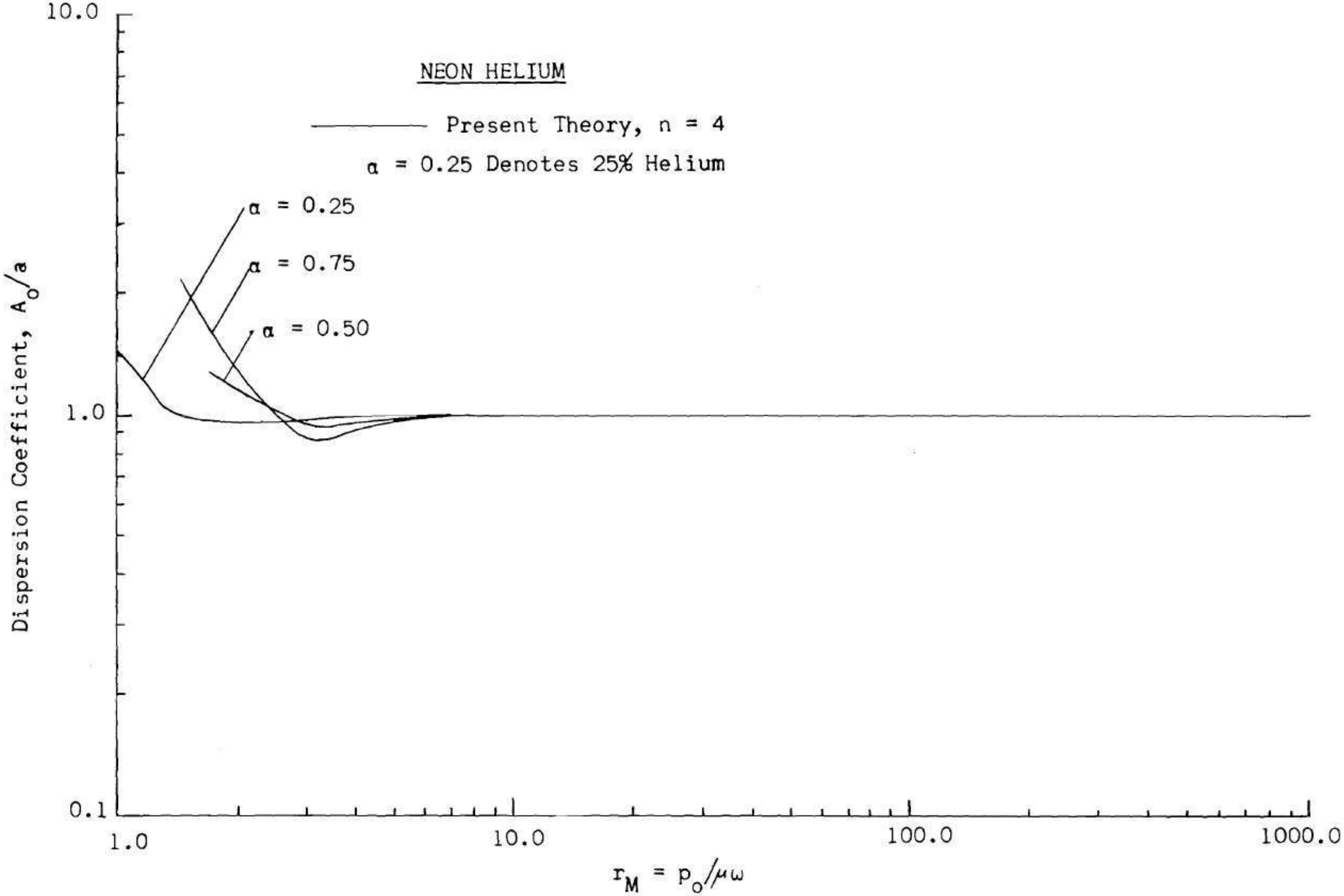

Figure 17. Comparison of Discrete Ordinate Solution for the Dispersion Coefficient in a Neon-Helium Mixture With Experiment.

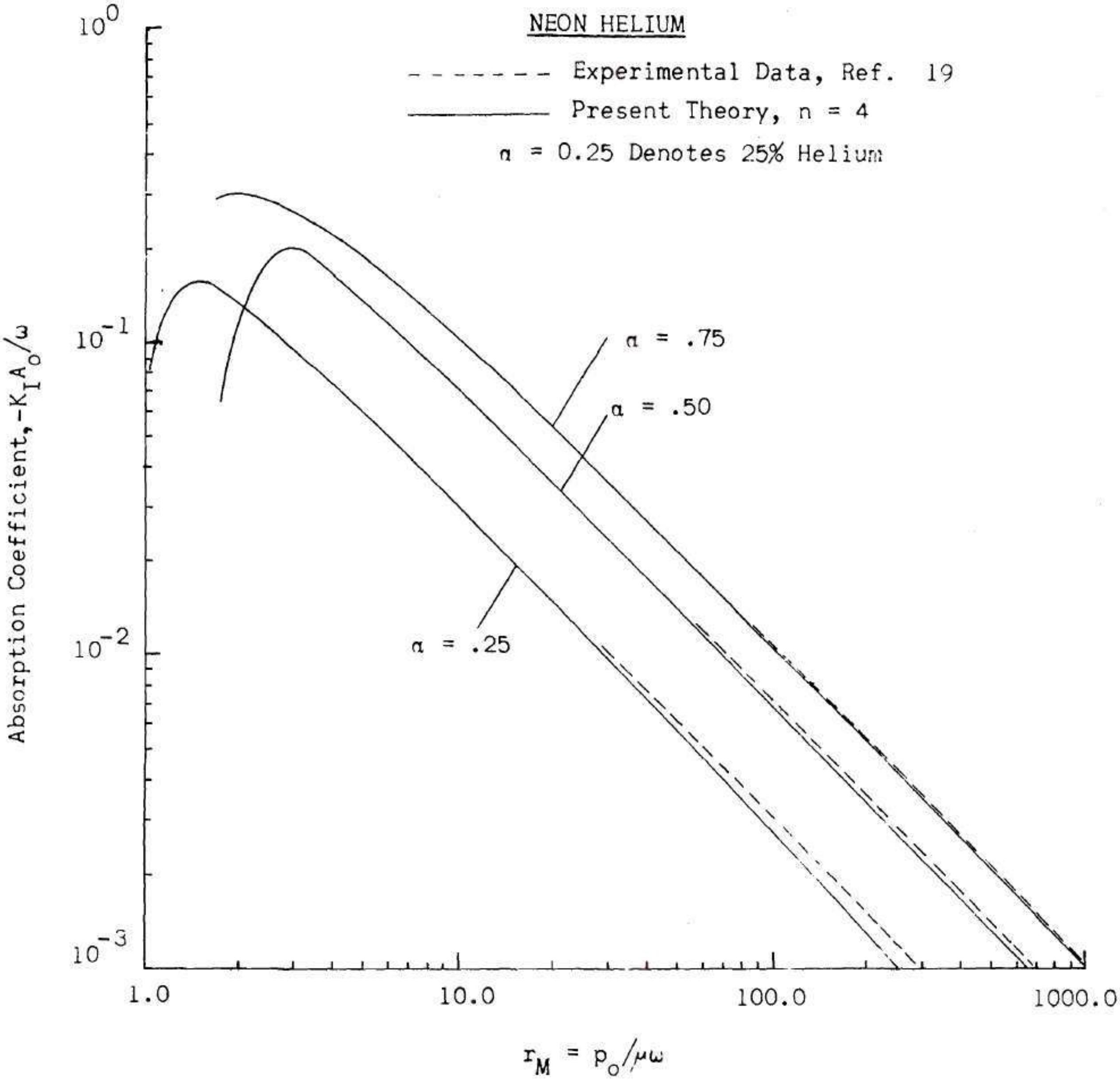

Figure 18. Comparison of Discrete Ordinate Solution for the Absorption Coefficient in a Neon-Helium Mixture With Experiment.

The calculations for Argon-Neon are shown in Figures 19 and 20. Since this mixture was not considered by Holmes and Tempest, no comparison with experiment could be made.

It is interesting to make a general comparison of these results with those for a single component gas. The dependence of the absorption coefficient on the ratio of collision frequency to sound frequency is very similar for both cases. The absorption coefficient goes to zero in the continuum regime and reaches a maximum for r_M of order 1.

In contrast to the results obtained for a single component gas, the dispersion coefficient for a mixture possesses a minimum for r_M approximately 2. As may be seen from Figures 15, 17 and 19, the minimum depends upon α and the mass ratio of the two elements. For Argon-Helium ($m^*/m = 10$) the minimums are approximately 0.65, 0.95 and 1.0 for $\alpha = 0.75$, 0.5 and 0.25, respectively (α refers to the number density ratio of the lighter element). A similar trend is evident for Neon-Helium ($m^*/m = 5$) and Argon-Helium ($m^*/m = 2$). Note also that the minimum is most pronounced when the mass ratio is large. Due to the lack of experimental data and theory in this regime, the validity of this result cannot be verified.

For r_M of order 1, it was found that some of the modes have an increasing amplitude. As noted by Greenspan [21], modes of this type occur for sound propagation in a simple gas if the theory is based upon the super-Burnett equations. No calculations were performed after these modes appeared.

In conclusion, it may be stated that the discrete ordinate method gives results which are in very good agreement with existing experimental

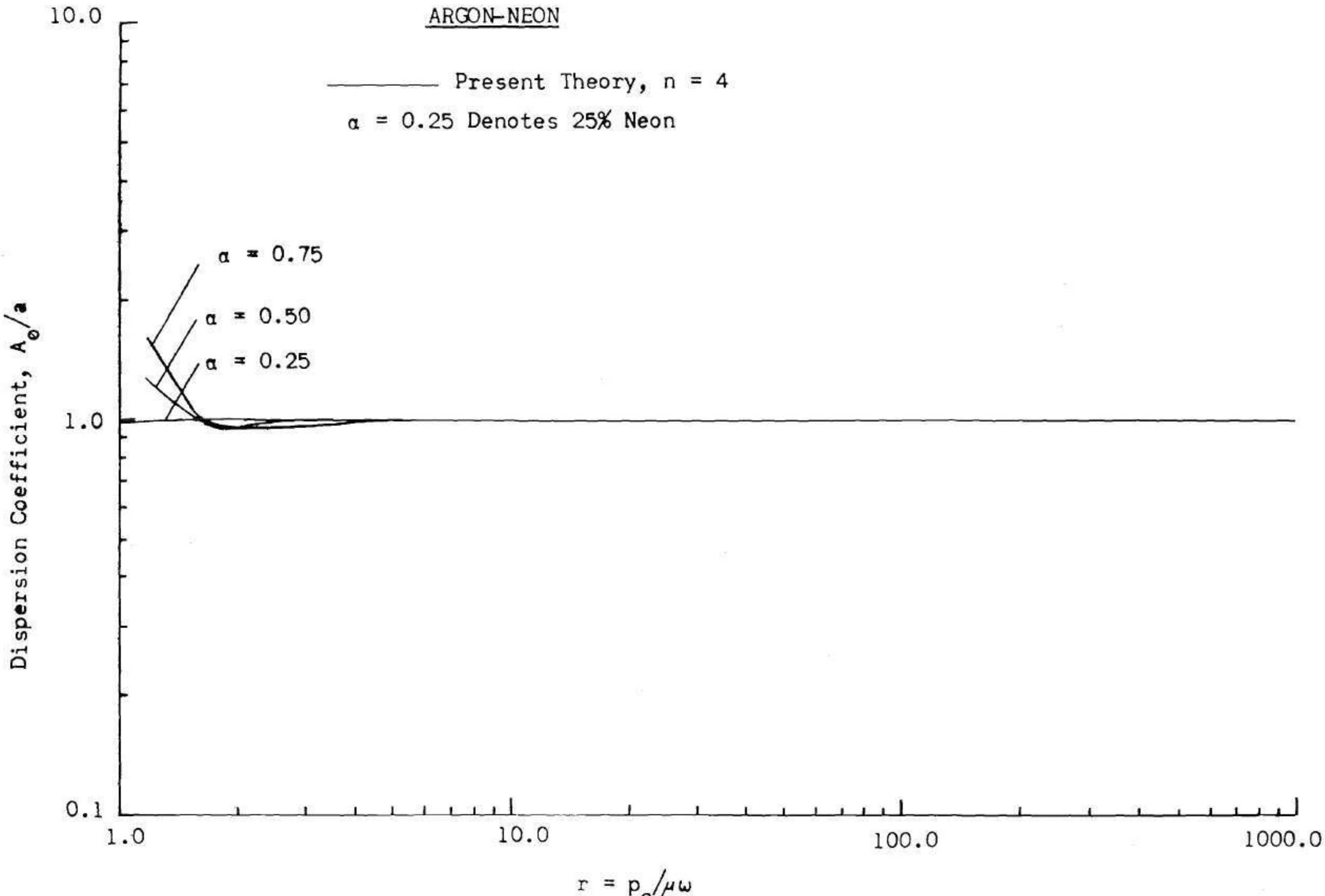

Figure 19. Discrete Ordinate Solution for the Dispersion Coefficient for an Argon-Neon Mixture.

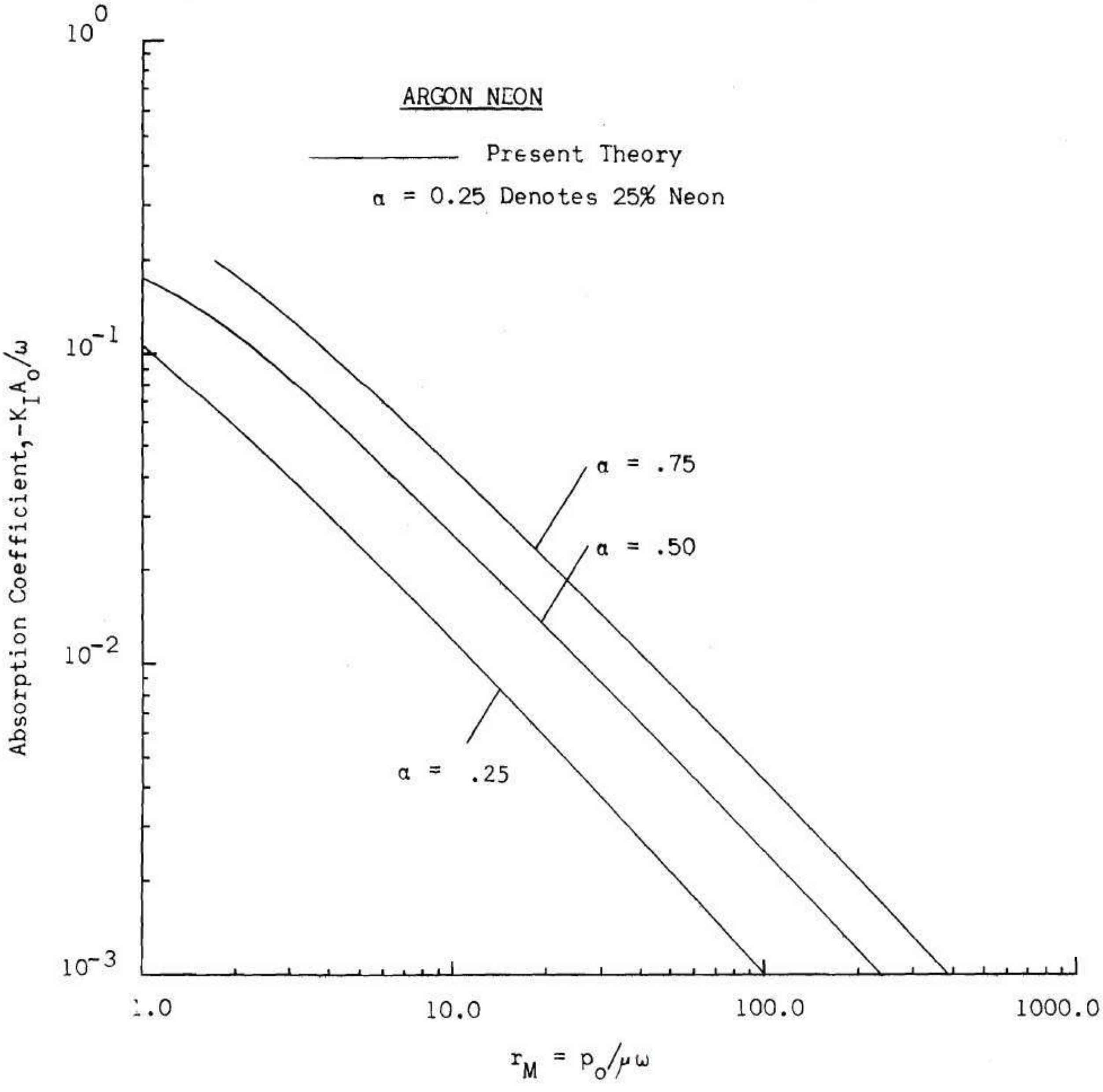

Figure 20. Discrete Ordinate Solution for the Absorption Coefficient in an Argon-Neon Mixture.

data. The solution predicts the correct dependence of the absorption and dispersion coefficients on the mixture ratio, and it has the correct "classical" limits in the continuum regime.

CHAPTER V

FREE SOUND WAVE PROPAGATION IN A MIXTURE OF GASES

Theoretical Formulation

In this chapter the propagation of an initial disturbance in an unbounded gas mixture is considered. The equilibrium state of the mixture is defined by the temperature T_o, the number densities n_o and n_o^* and the partial pressures p_o and p_o^*. Again the molecular masses of the components are denoted by m and m^*. At time $t = 0$, the gas is perturbed harmonically in the x-direction.

Discussion of the Model Equations

The analysis is again based upon the model equations proposed by Hamel [34]. In the last chapter it was shown that these equations could be written

$$\beta_o \frac{\partial \Phi}{\partial t} + c_x \frac{\partial \Phi}{\partial x} = \mathcal{L}(\Phi) \tag{1}$$

and

$$\beta_o \frac{\partial \Phi^*}{\partial t} + \left(\frac{m}{m^*}\right)^{1/2} c_x^* \frac{\partial \Phi^*}{\partial x} = \mathcal{L}(\Phi^*) \ , \tag{2}$$

where $\mathcal{L}(\Phi)$ and $\mathcal{L}(\Phi^*)$ denote the right hand sides of Equations (1) and (2) of Chapter IV, respectively.

Normal Mode Assumption

For this problem of free sound propagation in a mixture of two gases, it is assumed that the perturbations Φ and Φ^* have a space and time dependence of the form

$$\Phi(x,\bar{c},t) = \varphi(\bar{c})e^{-\sigma t+\hat{i}Kx} \tag{3}$$

and

$$\Phi^*(x,\bar{c}^*,t) = \varphi^*(\bar{c}^*)e^{-\sigma t+\hat{i}Kx} \ . \tag{4}$$

In this case, K is the wave number of the initial disturbance, and σ is the complex frequency. The problem is to calculate σ for a given initial wave number, K, and partial pressures, p_o and p_o^*.

By substituting Equations (3) and (4) into Equations (1) and (2), introducing Equations (9) through (14) of the previous chapter, and canceling the comming exponential term, one obtains the two coupled integral equations for the velocity dependent perturbations, φ and φ^*, given below:

$$\left(-\sigma\beta_o + \hat{i}Kc_x + \beta_o\left[\frac{p_o^*}{\mu^*}\left(\frac{\chi^*}{\chi^{**}}\right) + \frac{p_o}{\mu}\right]\right)\varphi(\bar{c}) = \mathcal{L}(\varphi) \tag{5}$$

and

$$\left(-\sigma\beta_o + \hat{i}\left(\frac{m}{m^*}\right)^{1/2}Kc_x^* + \beta_o\left[\frac{p_o}{\mu}\left(\frac{\chi^*}{\chi}\right) + \frac{p_o^*}{\mu^*}\right]\right)\varphi^*(\bar{c}^*) = \mathcal{L}(\varphi^*), \tag{6}$$

where $\mathcal{L}(\varphi)$ and $\mathcal{L}(\varphi^*)$ are defined by the right hand sides of Equations (19) and (20) of Chapter IV.

Transformation of the Integral Equations

To facilitate the solution of Equations (5) and (6) by the method of discrete ordinates, the (c_y, c_z) and (c_y^* and c_z^*) integrations are again eliminated by the use of the functions ψ, η, ψ^* and η^* which were defined in the previous chapter (Equations (21), (22), (23) and (24)). By following the procedure outlined in Chapter IV, one obtains

$$\sqrt{\pi}\left(-\sigma\beta_o + \hat{i}Kc_x + \beta_o\left[\frac{p_o^*}{\mu^*}\left(\frac{\chi^*}{\chi^{**}}\right) + \frac{p_o}{\mu}\right]\right)\psi(c_x) = \mathcal{L}(\psi) \tag{7}$$

$$\sqrt{\pi}\left(-\sigma\beta_o + \hat{i}Kc_x + \beta_o\left[\frac{p_o^*}{\mu^*}\left(\frac{\chi^*}{\chi^{**}}\right) + \frac{p_o}{\mu}\right]\right)\eta(c_x) = \mathcal{L}(\eta) \tag{8}$$

$$\sqrt{\pi}\left(-\sigma\beta_o + \hat{i}\left(\frac{m}{m^*}\right)^{1/2}Kc_x^* + \beta_o\left[\frac{p_o}{\mu}\left(\frac{\chi^*}{\chi}\right) + \frac{p_o^*}{\mu^*}\right]\right)\psi^*(c_x^*) = \mathcal{L}(\psi^*) \tag{9}$$

and

$$\sqrt{\pi}\left(-\sigma\beta_o + \hat{i}\left(\frac{m}{m^*}\right)^{1/2}Kc_x^* + \beta_o\left[\frac{p_o}{\mu}\left(\frac{\chi^*}{\chi}\right) + \frac{p_o^*}{\mu^*}\right]\right)\eta^*(c_x^*) = \mathcal{L}(\eta^*) \tag{10}$$

where $\mathcal{L}(\psi)$, $\mathcal{L}(\eta)$, $\mathcal{L}(\psi^*)$ and $\mathcal{L}(\eta^*)$ are defined by the right hand sides of Equations (25), (26), (27) and (28) of Chapter IV, respectively.

Free Sound Parameters for a Mixture

For the purpose of defining the free sound parameters r_M and r_M^*, a frequency $\omega_{Mo} = KA_o$, where $A_o = \left[\frac{5/3\ kT_o}{\alpha m + \alpha^* m^*}\right]^{1/2}$, must be introduced.

In terms of ω_{Mo}, r_M and r_M^* are written

$$r_M = \frac{p_o}{\mu\omega_{Mo}} \tag{11}$$

$$r_M^* = \frac{p_o^*}{\mu^*\omega_{Mo}} \quad . \tag{12}$$

One may show

$$r_M^* = \frac{1 - \alpha}{\alpha} \frac{\mu}{\mu^*} r_M , \tag{13}$$

so that once α is specified, only the parameter r_M need be varied.

The only other term to be introduced is a nondimensional frequency, $\hat{\sigma}_M$, which is defined by

$$\hat{\sigma}_M = \frac{\sigma}{\omega_{Mo}} . \tag{14}$$

Discrete Ordinate Solution

By approximating the integrals in Equations (7), (8), (9) and (10) by an appropriate quadrature and introducing Equations (11), (12) and (14), one obtains a system of equations of the form

$$\bar{\bar{A}} \begin{bmatrix} \bar{\psi} \\ \bar{\eta} \\ \bar{\psi}^* \\ \bar{\eta}^* \end{bmatrix} = \hat{\sigma}_M \bar{\bar{I}} \begin{bmatrix} \bar{\psi} \\ \bar{\eta} \\ \bar{\psi}^* \\ \bar{\eta}^* \end{bmatrix}$$

This system of equations will have a nontrivial solution only if $\hat{\sigma}_M$ is an eigenvalue of the matrix $\bar{\bar{A}}$, where $\bar{\bar{A}}$ has the elements

$$A_{\ell i} = \left(\hat{i} \left[\frac{6(\alpha m + \alpha^* m^*)}{5m} \right]^{1/2} c_\ell + r_M + \frac{\chi^*}{\chi^{**}} r_M^* \right) \delta_{\ell i} \tag{15}$$

$$- \frac{1}{\sqrt{\pi}} H_i \left((r_M^* \frac{\chi^*}{\chi^{**}} + r_M) \left[1 + 2 c_\ell c_i \right.\right.$$

$$\left. + \frac{2}{3}(c_\ell^2 - \frac{1}{2})(c_i^2 - \frac{3}{2}) \right] - 2\hat{m}^* \frac{\chi^*}{\chi^{**}} r_M^* c_\ell c_i$$

$$- \frac{4}{3}\hat{m}^*\hat{m}\frac{\chi^*}{\chi^{**}} r_M^*(c_\ell^2 - \frac{1}{2})(c_i^2 - \frac{3}{2})\Bigg)$$

$$A_{\ell,i+2n} = \frac{-1}{\sqrt{\pi}} H_i\Bigg(\frac{2}{3}(r_M^*\frac{\chi^*}{\chi^{**}} + r_M)(c_\ell^2 - \frac{1}{2}) - \frac{4}{3}\hat{m}^*\hat{m}r_M^*\frac{\chi^*}{\chi^{**}} \tag{16}$$

$$\cdot (c_\ell^2 - \frac{1}{2})\Bigg)$$

$$A_{\ell,i+4n} = \frac{-1}{\sqrt{\pi}} H_i\Bigg(2\hat{m}^*(\frac{m}{m^*})^{1/2} r_M^*\frac{\chi^*}{\chi^{**}} c_\ell c_i^* + \frac{4}{3}\hat{m}^*\hat{m}r_M^*\frac{\chi^*}{\chi^{**}} \tag{17}$$

$$\cdot (c_\ell^2 - \frac{1}{2})(c_i^{*2} - \frac{3}{2})\Bigg)$$

$$A_{\ell,i+6n} = \frac{-1}{\sqrt{\pi}} H_i \frac{4}{3}\hat{m}^*\hat{m}r_M^*\frac{\chi^*}{\chi^{**}}(c_\ell^2 - \frac{1}{2}) \tag{18}$$

$$A_{\ell+2n,i} = -\frac{1}{\sqrt{\pi}} H_i\Bigg((r_M^*\frac{\chi^*}{\chi^{**}} + r_M)\Big[1 + 2c_\ell c_i + \frac{2}{3}(c_\ell^2 + \frac{1}{2})(c_i^2 - \frac{3}{2})\Big]$$

$$-2\hat{m}^*\frac{\chi^*}{\chi^{**}} r_M^* c_\ell c_i - \frac{4}{3}\hat{m}^*\hat{m}\frac{\chi^*}{\chi^{**}} r_M^*(c_\ell^2 + \frac{1}{2})(c_i^2 - \frac{3}{2})\Bigg) \tag{19}$$

$$A_{\ell+2n,i+2n} = \left(\hat{i}\left[\frac{6(\alpha m + \alpha^* m^*)}{5m}\right]^{1/2} c_\ell + r_M + \frac{\chi^*}{\chi^{**}} r_M^*\right)\delta_{\ell i} \tag{20}$$

$$-\frac{1}{\sqrt{\pi}} H_i\Bigg(\frac{2}{3}(r_M^*\frac{\chi^*}{\chi^{**}} + r_M)(c_\ell^2 + \frac{1}{2})$$

$$- \frac{4}{3}\hat{m}^*\hat{m}r_M^*\frac{\chi^*}{\chi^{**}}(c_\ell^2 + \frac{1}{2})\Bigg)$$

$$A_{\ell+2n,i+4n} = \frac{-1}{\sqrt{\pi}} H_i\Bigg(2\hat{m}^*(\frac{m}{m^*})^{1/2} r_M^*\frac{\chi^*}{\chi^{**}} c_\ell c_i^* \tag{21}$$

$$+ \frac{4}{3}\hat{m}^*\hat{m}r_M^*\frac{\chi^*}{\chi^{**}}(c_\ell^2 + \frac{1}{2})(c_i^{*2} - \frac{3}{2})\Bigg)$$

$$A_{\ell+2n,i+6n} = \frac{-1}{\sqrt{\pi}} H_i \frac{4}{3} \hat{m}^*\hat{m}r_M^* \frac{\chi^*}{\chi^{**}}(c_\ell^2 + \frac{1}{2}) \tag{22}$$

$$A_{\ell+4n,i} = -\frac{1}{\sqrt{\pi}} H_i\left(2\hat{m}(\frac{m^*}{m})^{1/2} r_M \frac{\chi^*}{\chi} c_\ell^* c_i + \frac{4}{3} \hat{m}^*\hat{m}r_M \frac{\chi^*}{\chi}(c_\ell^{*2} - \frac{1}{2})(c_i^2 - \frac{3}{2})\right) \tag{23}$$

$$A_{\ell+4n,i+2n} = \frac{-1}{\sqrt{\pi}} H_i \frac{4}{3} \hat{m}^*\hat{m}r_M \frac{\chi^*}{\chi}(c_\ell^{*2} - \frac{1}{2}) \tag{24}$$

$$A_{\ell+4n,i+4n} = \left(\hat{i}\left[\frac{6(\alpha m + \alpha^* m^*)}{5m}\right]^{1/2} (\frac{m}{m^*})^{1/2} c_\ell^* + \frac{\chi^*}{\chi} r_M + r_M^*\right)\delta_{\ell i} - \frac{1}{\sqrt{\pi}} H_i\left((r_M \frac{\chi^*}{\chi} + r_M^*) \cdot \left[1 + 2c_\ell^* c_i^* + \frac{2}{3}(c_\ell^{*2} - \frac{1}{2})(c_i^{*2} - \frac{3}{2})\right] - 2\hat{m}r_M \frac{\chi^*}{\chi} c_\ell^* c_i^* - \frac{4}{3} \hat{m}^*\hat{m}r_M \frac{\chi^*}{\chi}(c_\ell^{*2} - \frac{1}{2})(c_i^{*2} - \frac{3}{2})\right) \tag{25}$$

$$A_{\ell+4n,i+6n} = \frac{-1}{\sqrt{\pi}} H_i\left(\frac{2}{3}(r_M \frac{\chi^*}{\chi} + r_M^*)(c_\ell^{*2} - \frac{1}{2}) - \frac{4}{3} \hat{m}^*\hat{m}r_M \frac{\chi^*}{\chi}(c_\ell^{*2} - \frac{1}{2})\right) \tag{26}$$

$$A_{\ell+6n,i} = -\frac{1}{\sqrt{\pi}} H_i\left(2\hat{m}(\frac{m^*}{m})^{1/2} r_M \frac{\chi^*}{\chi} c_\ell^* c_i^* + \frac{4}{3} \hat{m}^*\hat{m}r_M \frac{\chi^*}{\chi}(c_\ell^{*2} + \frac{1}{2})(c_i^2 - \frac{3}{2})\right) \tag{27}$$

$$A_{\ell+6n,i+2n} = \frac{-1}{\sqrt{\pi}} H_i \frac{4}{3} \hat{m}^*\hat{m}r_M \frac{\chi^*}{\chi}(c_\ell^{*2} + \frac{1}{2}) \tag{28}$$

$$A_{\ell+6n,i+4n} = \frac{-1}{\sqrt{\pi}} H_i \Big((r_M \frac{\chi^*}{\chi} + r_M^*)\Big[1 + 2c_\ell^* c_i^* + \frac{2}{3}(c_\ell^{*2} + \frac{1}{2})(c_i^{*2} - \frac{3}{2})\Big] - 2\hat{m} r_M \frac{\chi^*}{\chi} c_\ell^* c_i^* - \frac{4}{3} \hat{m}^* \hat{m} r_M \frac{\chi^*}{\chi}(c_\ell^{*2} + \frac{1}{2})(c_i^{*2} - \frac{3}{2}) \Big) \tag{29}$$

$$A_{\ell+6n,i+6n} = \Big(\hat{i} \Big[\frac{6(\alpha m + \alpha^* m^*)}{5m}\Big]^{1/2} (\frac{m}{m^*})^{1/2} c_\ell^* + \frac{\chi^*}{\chi} r_M + r_M^* \Big) \delta_{\ell i} - \frac{1}{\sqrt{\pi}} H_i \Big(\frac{2}{3}(r_M \frac{\chi^*}{\chi} + r_M^*) \cdot (c_\ell^{*2} + \frac{1}{2}) - \frac{4}{3} \hat{m}^* \hat{m} r_M \frac{\chi^*}{\chi}(c_\ell^{*2} + \frac{1}{2}) \Big), \tag{30}$$

where $\ell = 1, 2, \ldots, 2n$, $i = 1, 2, \ldots, 2n$ and n is the number of positive discrete points used in the quadrature approximation.

The problem is thus reduced to finding the eigenvalues of $\bar{\bar{A}}$ for given values of r_M and α for a particular gas mixture. See Chapter IV for a discussion of the computational procedure.

Discussion of Results

Solutions were obtained for Argon-Helium and Argon-Neon mixtures with number density ratios of 0.25, 0.50 and 0.75 (α again refers to the lighter element). The parameter r_M was varied from 1.0 to 100.0. All calculations are for n = 4.

The solution for an Argon-Helium mixture is presented in Figures 21 and 22. As may be seen from Figure 21, the solution for the absorption coefficient, $\frac{\sigma_R}{\omega_{Mo}}$, is very similar to the forced sound wave

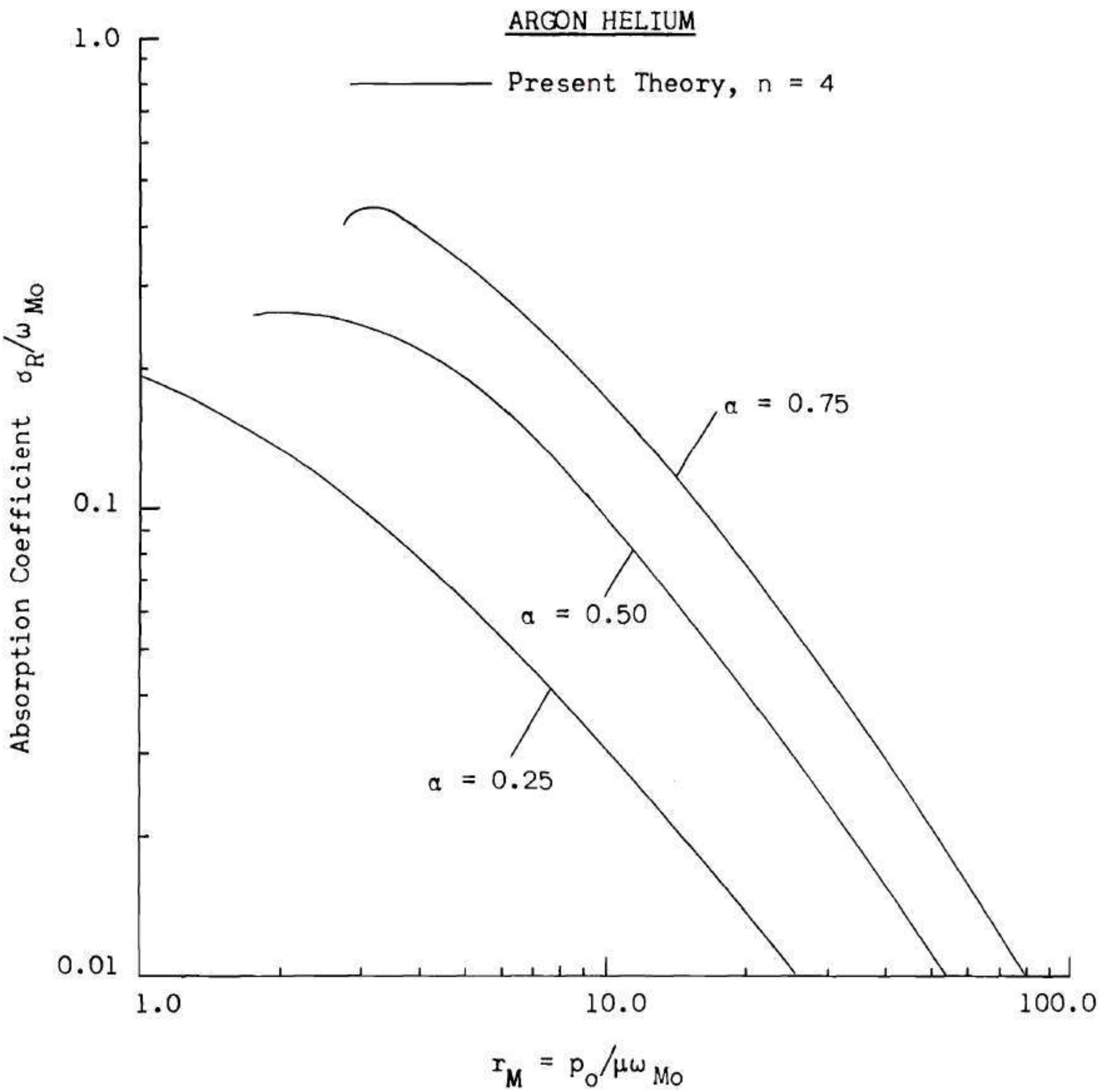

Figure 21. Absorption Coefficient for Free Sound Propagation in an Argon-Helium Mixture.

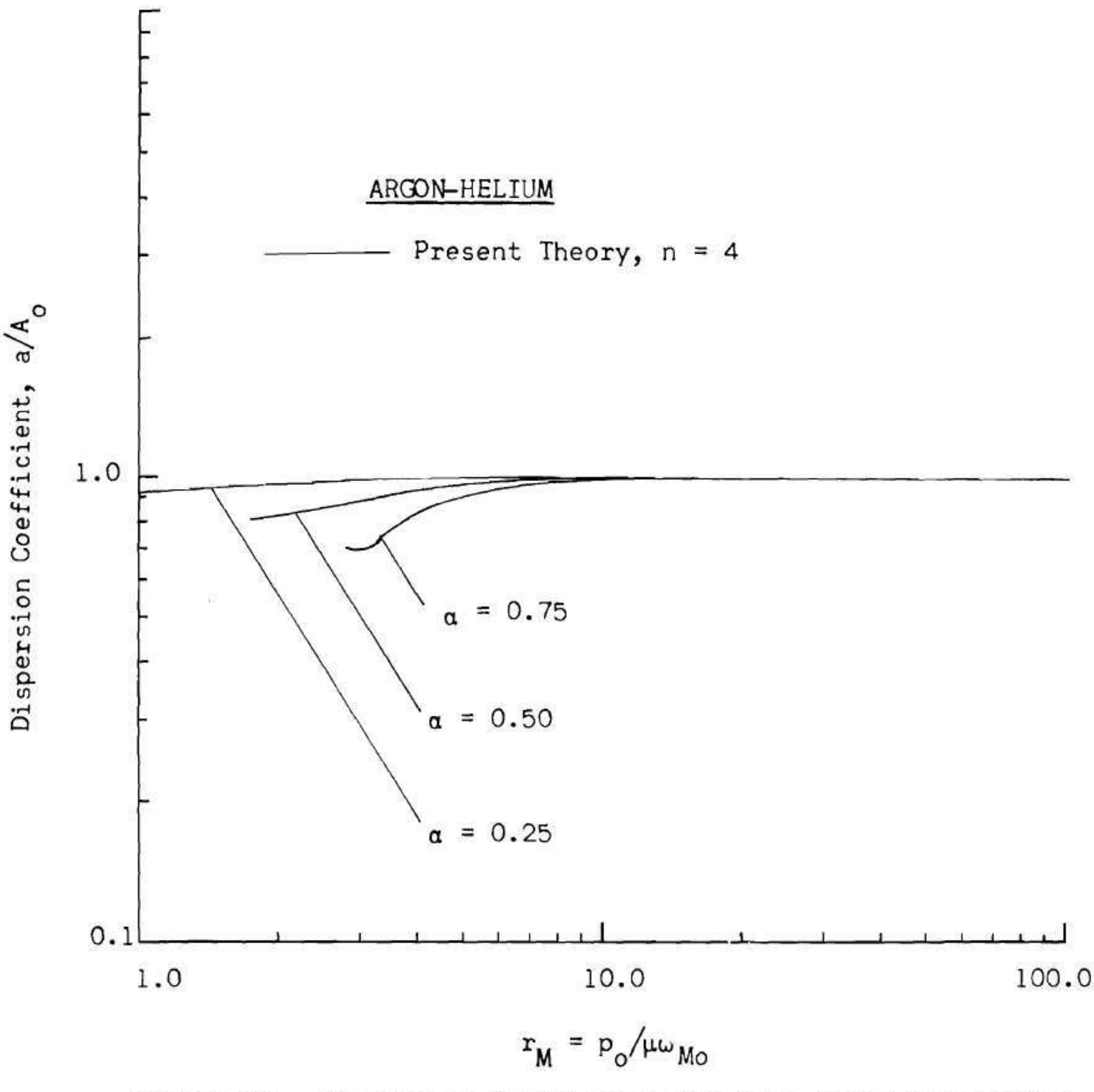

Figure 22. Dispersion Coefficient for Free Sound Propagation in an Argon-Helium Mixture.

absorption coefficient. The solution exhibits the same dependence on the number density ratio, α, and has the same classical limits for large values of r_M. Similar remarks apply to the solution for the dispersion coefficient, which is presented in Figure 22.

The solution for an Argon-Neon mixture is presented in Figures 23 and 24. The absorption and dispersion coefficients for this mixture have the same dependence on α and r_M as they do for an Argon-Helium mixture.

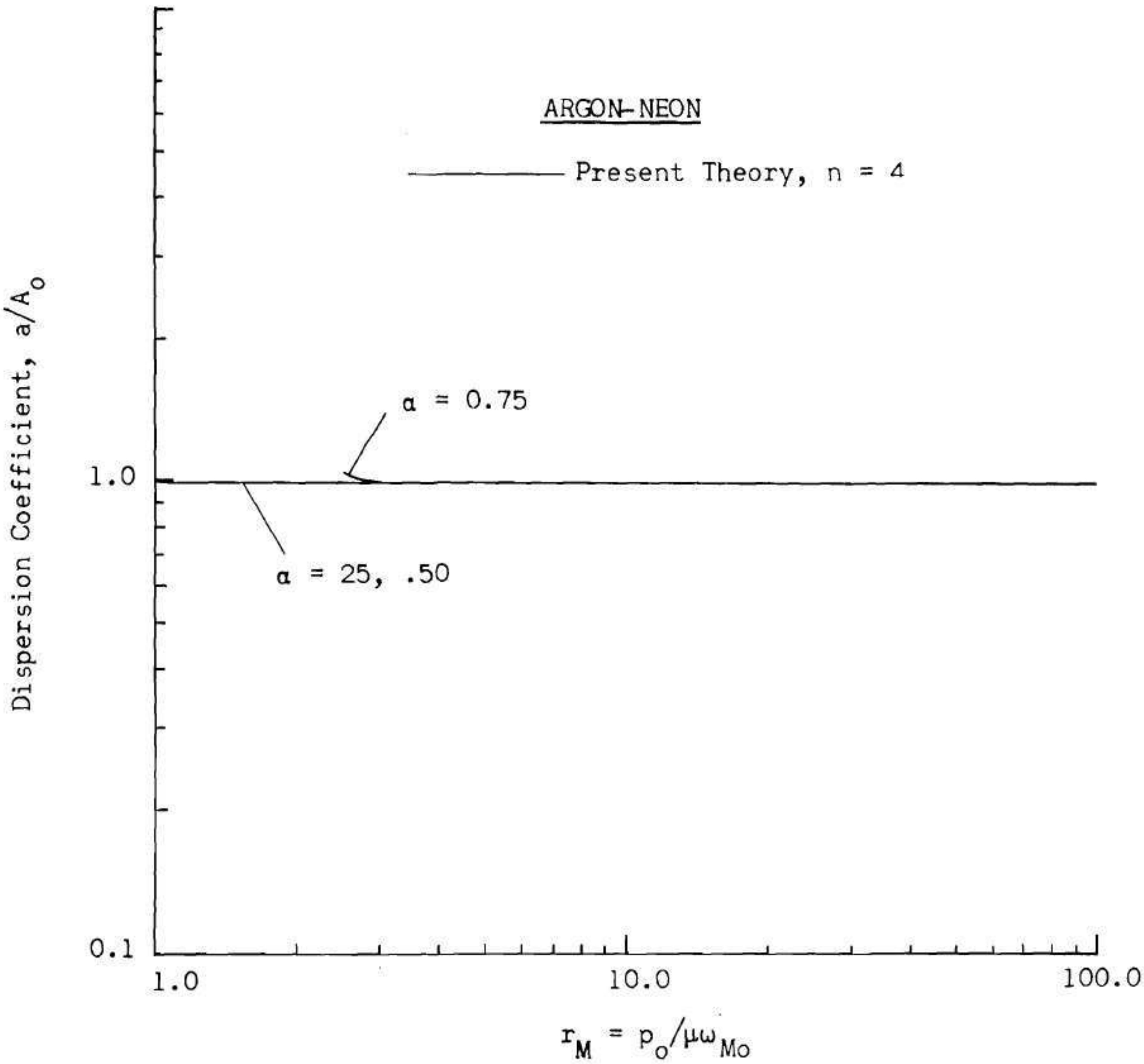

Figure 23. Dispersion Coefficient for Free Sound Propagation in an Argon-Neon Mixture.

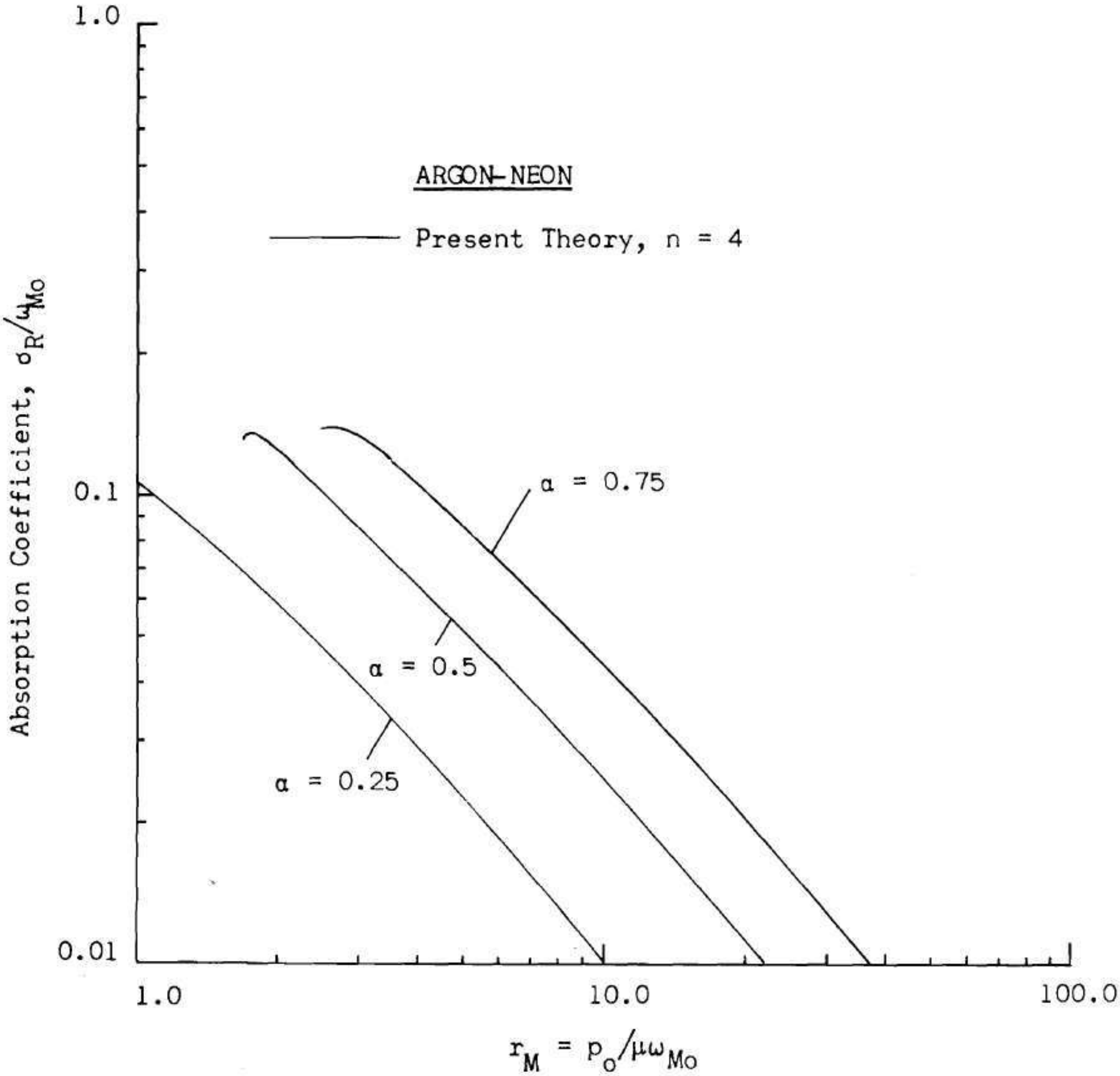

Figure 24. Absorption Coefficient for Free Sound Propagation in an Argon-Neon Mixture.

CHAPTER VI

PLASMA OSCILLATIONS

Theoretical Formulation

The purpose of this chapter is to demonstrate that oscillations in a system of charged particles may be treated by the discrete ordinate method. The physical problem is similar to free sound propagation, except that now the particles have an electric charge. At time $t = 0$ an infinite plasma consisting of electrons which move in a continuous positive background (a one-component plasma) is perturbed harmonically in the x-direction. The evolution of this initial perturbation in the absence of external electric and magnetic fields is the subject of this chapter. The analysis will be based upon the BGK and CT model equations.

BGK Model

Discussion of the Governing Equation. In this section the governing equation is the BGK model of the Boltzmann equation. This equation was discussed in detail in Chapter II, and thus only the modifications necessary for it to apply to a plasma are mentioned here. The most obvious modification is the introduction of the term $2\beta_o^2 \frac{\hat{e}E}{m} c_x$, which accounts for the long-range coulombic interaction between the electrons. This term results from linearizing the body force term, $\bar{a} \cdot \frac{\partial f}{\partial \bar{v}}$, which occurs in the Boltzmann equation. The linearization is straightforward and is not presented here. The second modification is the replacement of the term $\frac{1}{\Theta}$ by $\beta_o \nu$, where ν is a collision frequency (ν is denoted by

λ in Reference [11]). Recall that for a gas composed of neutral particles it was possible to relate Θ to the mean free path, ℓ. In the present case the collision frequency could be related to properties of the plasma (see, e.g., Reference [47]). However, this is not necessary for the purpose of this chapter. With these modifications, the BGK model becomes [11]

$$\beta_o \frac{\partial \Phi}{\partial t} + c_x \frac{\partial \Phi}{\partial x} + 2\beta_o^2 \frac{\hat{e}E}{m} c_x = \beta_o \nu \Big[- \Phi + \pi^{-3/2} \int e^{-c_1^2} \Phi(x,\bar{c}_1,t) d^3 c_1 \tag{1}$$

$$+ 2c_x \pi^{-3/2} \int_{-\infty}^{+\infty} e^{-c_1^2} c_{x_1} \Phi(x,\bar{c},t) d^3 c_1$$

$$+ \frac{2}{3}(c^2 - \frac{3}{2}) \pi^{-3/2} \int e^{-c_1^2} (c_1^2 - \frac{3}{2}) \Phi(x,\bar{c}_1,t) d^3 c_1 \Big] .$$

The electron charge is denoted by $\hat{e}$, and E is the electric field which results from variations in the electron number density. All other terms in Equation (1) have been defined previously.

For this simple, one-component plasma, the electric field, E, is found from the Poisson equation [11]

$$\frac{\partial E}{\partial x} = -4\pi\, n_o \hat{e}\, \pi^{-3/2} \int e^{-c_1^2} \Phi(x,\bar{c}_1,t) d^3 c_1 . \tag{2}$$

It is thus necessary to solve Equations (1) and (2) simultaneously.

Normal Mode Assumption. For this initial value problem, it is again appropriate to assume that the perturbation, Φ, has a space and time dependence of the form

$$\Phi(x,\bar{c}_1,t) = \varphi(\bar{c}) e^{-\sigma t + \hat{i} K x} \tag{3}$$

Recall that φ depends only on the molecular velocity, $\bar{c}$, K is the wave number of the initial disturbance and σ is the complex frequency, which is to be determined. Note that in this case $1/\sigma_R$ is the time required for the disturbance to decrease to 1/e of its initial value. The phase velocity of the disturbance is, of course, given by σ_I/K.

From Equations (2) and (3), one may easily show that the electric field, E, is given by

$$E = \frac{4\hat{e}n_o \hat{i}}{(\pi)^{1/2}K} e^{-\sigma t + \hat{i}Kx} \int e^{-c_1^2} \varphi(\bar{c}_1) d^3c_1 \quad . \tag{4}$$

By substituting Equations (3) and (4) into Equation (1) and canceling the common exponential term, one obtains the governing equation for the velocity dependent perturbation, $\varphi(\bar{c})$, written below:

$$-\sigma\beta_o\varphi(\bar{c}) + \hat{i}Kc_x\varphi(\bar{c}) + \hat{i}\,\frac{2\beta_o^2 c_x}{K}\left(\frac{4\pi\hat{e}^2 n_o}{m}\right)\pi^{-3/2}\int e^{-c_1^2}\varphi(\bar{c}_1)d^3c_1 \tag{5}$$

$$= \beta_o\nu\Big[-\varphi(\bar{c}) + \pi^{-3/2}\int e^{-c_1^2}\varphi(\bar{c}_1)d^3c_1 + 2c_x\pi^{-3/2}\int e^{-c_1^2}$$

$$\cdot\, c_{x_1}\varphi(\bar{c}_1)d^3c_1 + \frac{2}{3}(c^2 - \frac{3}{2})\pi^{-3/2}\int e^{-c_1^2}(c_1^2 - \frac{3}{2})\varphi$$

$$\cdot\,(\bar{c}_1)d^3c_1\Big] \; .$$

This equation differs from the free sound governing equation only in the third term on the left side.

Transformation of the Integral Equation. The velocity integrations perpendicular to the direction of propagation are eliminated by use of the transformation described in Chapter II. The results of this transformation are

$$\sqrt{\pi}\left[-\sigma\beta_o + \hat{i}Kc_x + \beta_o\nu\right]\psi(c_x) \tag{6}$$

$$= \beta_o\nu\left[(1 - \hat{i}\,\frac{2\beta_o^2 c_x}{K\beta_o\nu}\,(\frac{4\pi\hat{e}^2 n_o}{m}))\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\psi(c_{x_1})dc_{x_1}\right.$$

$$+ 2c_x\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\, c_{x_1}\,\psi(c_{x_1})dc_{x_1}$$

$$+ \frac{2}{3}(c_x^2 - \frac{1}{2})\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\left[\eta(c_{x_1}) + (c_{x_1}^2 = \frac{3}{2})\right.$$

$$\left.\left.\cdot\ \psi(c_{x_1})\right]dc_{x_1}\right]$$

and

$$\sqrt{\pi}\left[-\ \sigma\beta_o + \hat{i}Kc_x + \beta_o\nu\right]\eta(c_x) \tag{7}$$

$$= \beta_o\nu\left[(1 - \hat{i}\,\frac{2\beta_o^2 c_x}{K\beta_o\nu}\,(\frac{4\pi\hat{e}^2 n_o}{m}))\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\psi(c_{x_1})dc_{x_1}\right.$$

$$+ 2c_x\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\, c_{x_1}\psi(c_{x_1})dc_{x_1}$$

$$+ \frac{2}{3}(c_x^2 + \frac{1}{2})\int_{-\infty}^{+\infty} e^{-c_{x_1}^2}\left[\eta(c_{x_1}) + (c_{x_1}^2 - \frac{3}{2})\right.$$

$$\left.\left.\cdot\ \psi(c_{x_1})\right]dc_{x_1}\right]\ .$$

where

$$\psi(c_x) = \pi^{-3/2} \int_{-\infty}^{+\infty} dc_y \int_{-\infty}^{+\infty} dc_z e^{-c_y^2 - c_z^2} \varphi(\bar{c}) , \tag{8}$$

$$\eta(c_x) = \pi^{-3/2} \int_{-\infty}^{+\infty} dc_y \int_{-\infty}^{+\infty} dc_z e^{-c_y^2 - c_z^2} (c_y^2 + c_z^2) \varphi(\bar{c}) \tag{9}$$

Plasma Oscillation Parameters. It is again possible to introduce convenient nondimensional parameters. The expression $(4\pi\hat{e}^2 n_o/m)$ is the square of the plasma frequency, ω_p, i.e.,

$$\omega_p = \left(\frac{4\pi\hat{e}^2 n_o}{m}\right)^{1/2} . \tag{10}$$

By using Equation (10), one may define nondimensional frequencies, $\hat{\sigma}_p$ and $\hat{\nu}$, by

$$\hat{\sigma}_p = \frac{\sigma}{\omega_p} \tag{11}$$

$$\hat{\nu} = \frac{\nu}{\omega_p} \tag{12}$$

One other term is introduced, a nondimensional wave number $\hat{K}_p$, which is defined by

$$\hat{K}_p = \frac{K\left(\frac{kT_o}{m}\right)^{1/2}}{\omega_p} \tag{13}$$

This choice was made in order to compare the discrete ordinate solution with that of Reference [11]. Note that $(kT_o/m)^{1/2}/\omega_p$ is simply the Debye length.

Discrete Ordinate Solution. By approximating the integrals in Equations (6) and (7) by finite summations, introducing Equations (11), (12) and (13), and rearranging, one obtains

$$(\hat{i}\sqrt{2}\,c_\ell \hat{K}_p + \hat{\nu})\psi_\ell + \frac{1}{\sqrt{\pi}} \sum_{i=1}^{2n} H_i\Bigg(\hat{i}\,\frac{\sqrt{2}\,c_\ell}{\hat{K}_p} \tag{14}$$

$$- \hat{\nu}\left[1 + 2c_\ell c_i + \frac{2}{3}(c_\ell^2 - \frac{1}{2})(c_i^2 - \frac{3}{2})\right]\Bigg)\psi_i$$

$$- \frac{1}{\sqrt{\pi}} \sum_{i=1}^{2n} H_i\,\frac{2}{3}\,\hat{\nu}(c_\ell^2 - \frac{1}{2})\,\eta_i = \hat{\sigma}_p\psi_\ell$$

and

$$\frac{1}{\sqrt{\pi}} \sum_{i=1}^{2n} H_i\Bigg(\hat{i}\,\frac{\sqrt{2}\,c_\ell}{\hat{K}_p} - \hat{\nu}\left[1 + 2c_\ell c_i + \frac{2}{3}(c_\ell^2 + \frac{1}{2})(c_i^2 - \frac{3}{2})\right]\Bigg)\psi_i \tag{15}$$

$$+ (\hat{i}\sqrt{2}\,c_\ell \hat{K}_p + \hat{\nu})\eta_\ell - \frac{1}{\sqrt{\pi}} \sum_{i=1}^{2n} H_i\Bigg(\frac{2}{3}\,\hat{\nu}$$

$$\cdot\,(c_\ell^2 + \frac{1}{2})\Bigg)\eta_i = \hat{\sigma}_p\eta_\ell$$

Equations (14) and (15) are slightly different from any encountered previously in that there are two free parameters to be specified, $\hat{K}_p$ and $\hat{\nu}$. $\hat{K}_p$ is the nondimensional wave number of the initial disturbance, and

$\hat{\sigma}_p$ is a nondimensional collision frequency. The problem is to calculate $\hat{\sigma}_p$ for given values of $\hat{K}_p$ and $\hat{\nu}$. As in previous chapters, $\hat{\sigma}_p$ is found by calculating the eigenvalues of the matrix $\bar{\bar{A}}$, where in this case $\bar{\bar{A}}$ has the elements

$$A_{\ell i} = (\hat{i}\sqrt{2}\,c_\ell \hat{K}_p + \hat{\nu})\delta_{\ell i} + \frac{1}{\sqrt{\pi}} H_i\left(\hat{i}\,\frac{\sqrt{2}\,c_\ell}{\hat{K}_p} - \hat{\nu}\left[1 + 2c_\ell c_i + \frac{2}{3}(c_\ell^2 - \frac{1}{2})(c_i^2 - \frac{3}{2})\right]\right) \tag{16}$$

$$A_{\ell, i+2n} = -\frac{1}{\sqrt{\pi}} H_i \frac{2}{3}\hat{\nu}(c_\ell^2 - \frac{1}{2}) \tag{17}$$

$$A_{\ell+2n, i} = \frac{1}{\sqrt{\pi}} H_i\left(\hat{i}\,\frac{\sqrt{2}\,c_\ell}{\hat{K}_p} - \hat{\nu}\left[1 + 2c_\ell c_i + \frac{2}{3}(c_\ell^2 + \frac{1}{2})(c_i^2 - \frac{3}{2})\right]\right) \tag{18}$$

$$A_{\ell+2n, i+2n} = (\hat{i}\sqrt{2}\,c_\ell \hat{K}_p + \hat{\nu})\delta_{\ell i} - \frac{1}{\sqrt{\pi}} H_i \cdot \frac{2}{3}\hat{\nu}(c_\ell^2 + \frac{1}{2}) \ , \tag{19}$$

where $\ell = 1, 2, \ldots, 2n$, $i = 1, 2, \ldots, 2n$, and n is the number of positive discrete velocity points.

A discussion of the results for the BGK model is deferred until after the formulation for the CT model.

CT Model

Discussion of the Governing Equation. As was the case for the BGK model, the CT equation must be modified to account for the coulombic interaction between electrons. With this modification, the CT model becomes

$$\beta_o \frac{\partial \Phi}{\partial t} + c_x \frac{\partial \Phi}{\partial x} + 2\beta_o^2 \frac{\hat{e}E}{m} c_x \tag{20}$$

$$= \beta_o \frac{2}{\lambda + 2} \nu \Big(- \Phi + \pi^{-3/2} \int e^{-c_1^2} \Phi(x,\bar{c}_1,t)d^3c_1$$

$$+ 2c_x \pi^{-3/2} \int e^{-c_1^2} c_{x_1} \Phi(x,\bar{c}_1,t)d^3c_1$$

$$+ \frac{2}{3}(c^2 - \frac{3}{2})\pi^{-3/2} \int e^{-c_1^2}(c_1^2 - \frac{3}{2})\Phi(x,\bar{c}_1,t)d^3c_1$$

$$- \lambda\Big[c_x^2 \pi^{-3/2} \int e^{-c_1^2} c_{x_1} \Phi(x,\bar{c}_1,t)d^3c_1$$

$$+ \frac{1}{2}(c_y^2 + c_z^2) \pi^{-3/2} \int e^{-c_1^2}(c_{y_1}^2 + c_{z_1}^2)\Phi(x,\bar{c}_1,t)d^3c_1\Big]$$

$$+ \frac{\lambda}{2}\Big[c^2(\pi^{-3/2} \int e^{-c_1^2}\Phi(x,\bar{c}_1,t)d^3c_1 + \frac{2}{3}$$

$$\cdot \pi^{-\frac{2}{3}} \int e^{-c_1^2}(c_1^2 - \frac{3}{2})\Phi(x,\bar{c}_1,t)d^3c_1)\Big]\Big),$$

where E is given by Equation (2).

Normal Mode Formulation. The formulation from this point is identical to the formulation for the BGK model. By substituting Equations (3) and (4) into Equation (20), one obtains an equation for $\varphi(\bar{c})$. This equation may be transformed as before to yield a coupled pair of equations for ψ and η. If the integrals in this pair of equations are approximated by finite summations, one obtains a system of equations of the form

$$\bar{\bar{A}} \begin{pmatrix} \bar{\psi} \\ \bar{\eta} \end{pmatrix} = \sigma_p \bar{\bar{I}} \begin{pmatrix} \bar{\psi} \\ \bar{\eta} \end{pmatrix} ,$$

where $\bar{\bar{A}}$ has the elements

$$A_{\ell i} = (\hat{i}\sqrt{2}\, c_\ell \hat{K}_p + \frac{2\hat{\nu}}{\lambda + 2})\delta_{\ell i} + \hat{i}\frac{\sqrt{2}\, c_\ell}{\sqrt{\pi}\, \hat{K}_p} H_i \tag{21}$$

$$- \frac{2\hat{\nu}}{\sqrt{\pi}\,(\lambda + 2)} H_i \Big(1 + 2c_\ell c_i$$

$$+ \frac{2}{3}(c_i^2 - \frac{1}{2})(c_i^2 - \frac{3}{2}) - \frac{2}{3}\lambda(c_\ell^2 - \frac{1}{2})c_i^2 \Big)$$

$$A_{\ell, i+2n} = \frac{-2\hat{\nu}}{\sqrt{\pi}\,(\lambda + 2)} H_i \Big(\frac{2}{3}(c_\ell^2 - \frac{1}{2}) + \frac{1}{3}\lambda(c_\ell^2 - \frac{1}{2}) \Big) \tag{22}$$

$$A_{\ell+2n, i} = \hat{i}\frac{\sqrt{2}\, c_\ell}{\sqrt{\pi}\, \hat{K}_p} H_i - \frac{2\hat{\nu}}{\sqrt{\pi}\,(\lambda + 2)} H_i \Big(1 + 2c_\ell c_i \tag{23}$$

$$+ \frac{2}{3}(c_\ell^2 + \frac{1}{2})(c_i^2 - \frac{3}{2}) - \frac{2}{3}\lambda(c_\ell^2 - 1)c_i^2 \Big)$$

$$A_{1+2n, i+2n} = (\hat{i}\sqrt{2}\, c_\ell \hat{K}_p + \frac{2\hat{\nu}}{\lambda + 2})\delta_{\ell i} \tag{24}$$

$$- \frac{2\hat{\nu}}{\sqrt{\pi}\,(\lambda + 2)} H_i \Big(\frac{2}{3}(c_\ell^2 + \frac{1}{2}) + \frac{1}{3}\lambda(c_\ell^2 - 1) \Big) ,$$

$\ell = 1, 2, \ldots, 2n$, $i = 1, 2, \ldots, 2n$. Once n and λ are specified, $\bar{\bar{A}}$ may be calculated from Equations (21), (22), (23), and (24).

Discussion of Results

The eigenvalues of $\bar{\bar{A}}$ for both the BGK and CT model equations were calculated for n = 4 to 8. The parameter $\hat{K}_p$ was varied from 0.1 to 0.5, and the nondimensional collision frequency, $\hat{\nu}$, was varied from 0 to 5. All calculations for the CT model are for $\lambda = 1.0$.

The results of these calculations are shown in Figures 25 and 26. In these figures, the absorption coefficient, σ_R/ω_p, and the dispersion coefficient, σ_I/ω_p, are plotted against the nondimensional collision frequency, $\hat{\nu}$.

In Figure 25, the BGK and CT solutions for the dispersion coefficient are compared with the results predicted by Equation 83 of Reference [11]. Also shown on this figure are the points at which the discrete ordinate solutions cease to converge. As may be seen, the discrete ordinate solutions converge for $\hat{K}_p$ = 0.1, 0.2 and 0.3. For $\hat{K}_p$ = 0.4 and 0.5, there exists values of $\hat{\nu}$ below which convergence is no longer obtained. This value of $\hat{\nu}$ increases as $\hat{K}_p$ increases. Note that there is very good agreement with the theory of Reference [11]. For large values of $\hat{\nu}$, it may be shown [11]

$$\frac{\sigma_I}{\omega_p} \sim 1 + A\hat{K}_p^2$$

where A depends on $\hat{\nu}$. The discrete ordinate solution clearly exhibits the dependence for $\hat{\nu} > 3$. It may also be seen that the BGK and CT models predict nearly the same results.

The absorption coefficient predicted by the discrete ordinate solution is compared with the theory of Reference [11] in Figure 26. As

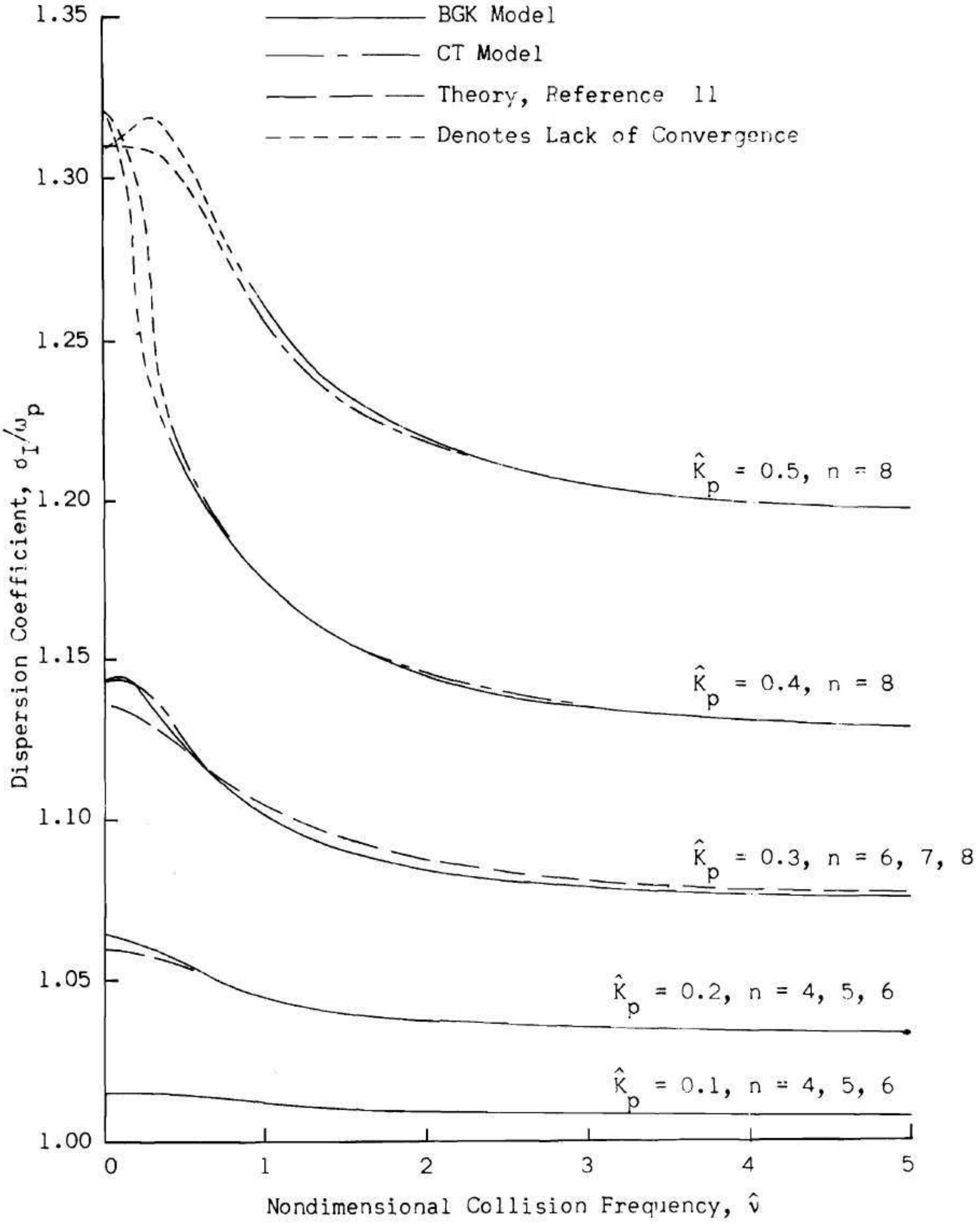

Figure 25. Dispersion Coefficient for One-Component Plasma.

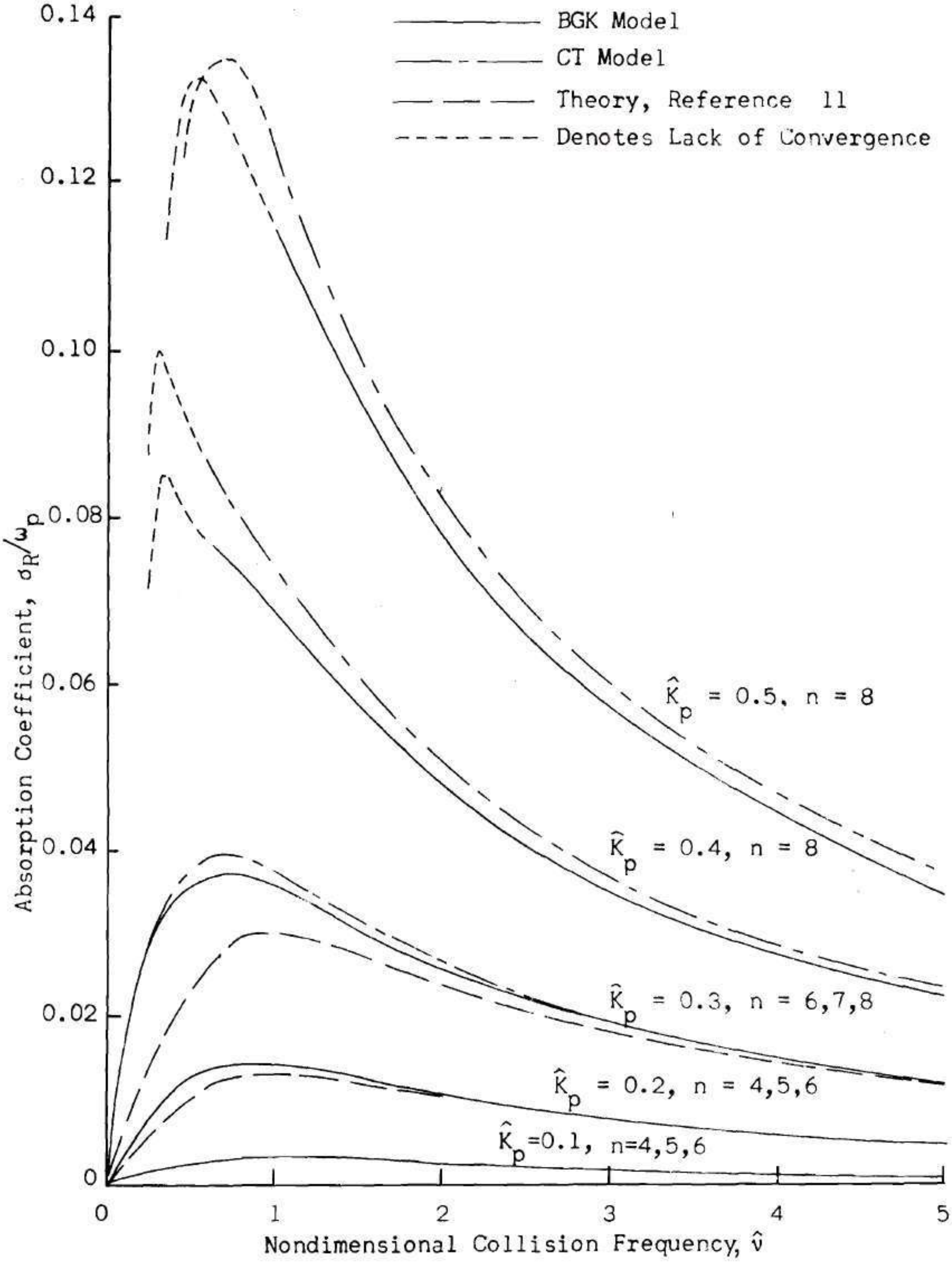

Figure 26. Absorption Coefficient for One-Component Plasma.

may be seen, the comparison is excellent for $\hat{K}_p = 0.1$ for all values of $\hat{\nu}$. For $\hat{K}_p = 0.2$, good agreement exists at large values of $\hat{\nu}(\hat{\nu} > 2)$. For $\hat{\nu} < 2$, the maximum deviation is approximately 10 per cent. For $\hat{K}_p = 0.3$, the discrete ordinate theory differs from the theory of Reference [11] by as much as twenty per cent for $\hat{\nu} < 1.5$. It should be noted that Equation 83 of Reference [11] is strictly valid only for $\hat{K}_p \ll 1$. Since the discrete ordinate solutions converged for $\hat{K}_p = 0.3$, it is felt they are correct. For large values of $\hat{\nu}$ it may be shown that

$$\frac{\sigma_R}{\omega_p} \sim B\hat{K}_p^2 \quad ,$$

where B depends on $\hat{\nu}$. That the discrete ordinate theory has this dependence may be seen by examining the solutions for $\hat{\nu} > 3$. Again the BGK and CT solutions are in very close agreement.

CHAPTER VII

DISCUSSION AND CONCLUSIONS

The method of discrete ordinates has been applied to the problem of wave propagation in simple monatomic gases, gas mixtures, and plasmas. This method of solution has been used with great success for other gas dynamics problems (e.g., Couette flow, Rayleigh flow, etc.), however, it has not been applied to the problem of wave propagation before. For this reason, it was necessary to determine the accuracy of the discrete ordinate method for this problem. Because good experimental data and many theories exist for the problem of forced sound propagation in a monatomic gas, this problem was chosen to calibrate the discrete ordinate method. As a further check, free sound propagation in a simple monatomic gas was also considered. After the accuracy of the discrete ordinate method was established, the problems of forced and free sound wave propagation in a binary mixture were solved. Finally, it was demonstrated that the discrete ordinate method may be used in the study of plasma oscillation.

Throughout this thesis, models of the Boltzmann collision integral were used. These models made it possible to transform the single integrodifferential equation involving integrations over velocity space (c_x, c_y, c_z) into a coupled pair of integrodifferential equations which contained a single integration over c_x. In general, a transformation of this type will not be possible if the full collision integral is retained. However, the discrete ordinate method may still be used.

The results of this investigation may be summarized in the following conclusions:

1. If a normal mode assumption regarding the space and time dependence of the perturbation distribution function is made, the discrete ordinate technique yields results which are very similar to the polynomial expansion method. This was demonstrated for both free and forced sound wave propagation. For forced sound wave propagation in a monatomic gas, it was found that the $n = 8$ discrete ordinate solution was comparable to the twenty-six term solution of Pekeris, et al. [6]. The $n = 8$ solution was also comparable to a Grad 25 moment approximation [7]. The free sound wave solution was similar to the Grad 13 moment theory [16]. The $n = 8$ normal mode solution was in good agreement with the transformation theories for $r > 1$. Excellent agreement with the experimental data of Meyer and Sessler [15] was obtained for $r > 1$.

2. A solution valid in a highly rarefied gas ($r \ll 1$) may be obtained by solving for the perturbation distribution function explicitly. The solution is in good agreement with experimental data [15] and other theories [7, 9, 10, 12] for $r < 0.1$. This method of solution also yields the transient development of the perturbation pressure field. It was found that the approach to a steady state is quite rapid.

3. A solution to the problem of forced sound propagation in a binary gas mixture was obtained which has the correct "clasical" limits for $r \gg 1$ and has the correct dependence on the concentrations of the two components. The solution was in excellent agreement with the experimental data of Holmes and Tempest [19] for both Argon-Helium and Neon-Helium mixtures. A solution was also obtained for an Argon-Neon mixture.

4. The problem of free sound wave propagation in a binary mixture was also solved. There is no known theory with which this solution may be compared, however, the solution appears reasonable.

5. It is possible to use the method of discrete ordinates in the study of plasma oscillations. This was demonstrated by solving the problem of longitudinal electrostatic oscillations in a one-component plasma. The solution obtained was in good agreement with the theory of Bhatnagar, Gross and Krook [11]. It was found that the results predicted by the BGK and CT models were in very close agreement.

6. A major advantage of the discrete ordinate method is the ease with which higher approximations may be obtained. It appears that the method has great flexibility and may be applied to many practical problems.

APPENDIX

COMPUTATIONAL PROCEDURE FOR HIGHLY RAREFIED SOLUTION

The purpose of this appendix is to illustrate the procedure used to obtain the absorption and dispersion coefficients in the highly rarefied regime.

The envelope of the perturbation pressure and two typical pressure distributions are shown in Figure 49. Since the absorption and dispersion are functions of position, it is necessary to analyze the pressure field at a particular value of ξ. For the experiment of Meyer and Sessler, the location is determined from

$$\frac{x}{\ell} = 9.7r \ , \tag{1}$$

where $r = p_o/\mu\omega$. Knowing x/ℓ, one may readily obtain ξ from the equation

$$\frac{x}{\ell} = \frac{.565}{\varepsilon^*} \ \ln \frac{1 + \xi}{1 + \xi} \tag{2}$$

For the solution presented in Figure 49, $\varepsilon^* = 1.0$ and $r = 0.01$. Using these values of ε^* and r in Equations (1) and (2) yields

$$\xi = .086 \ ,$$

the value of ξ at which the pressure field is to be analyzed.

The absorption coefficient, K_I, may be obtained from

$$K_I = \frac{-d \ln p_{xx}}{dx} \approx -\frac{1}{\ell} \frac{\Delta \ln p_{xx}}{\Delta (\frac{x}{\ell})} \quad ,$$

where p_{xx} denotes the maximum perturbation pressure at a given point. Using the pressure peaks at ξ = 0.08 and 0.087, one obtains from Equation (3) the result

$$\ell K_I = 21.1 \quad . \tag{4}$$

Since $\omega\Theta\beta_o = \frac{16}{5\pi}\frac{1}{r}$ and $\beta_o = \sqrt{5/6}\,\frac{1}{a_o}$, one may write

$$\ell = \frac{.972}{r} \frac{a_o}{\omega} \tag{5}$$

For $r = 0.01$, $\ell = 97.2\ a_o/\omega$, so that

$$\frac{K_I a_o}{\omega} = -0.22 \tag{6}$$

The phase velocity of the wave may be determined from

$$a \approx \frac{\Delta x}{\Delta t} = \ell\omega \frac{\Delta (\frac{x}{\ell})}{\Delta \tau} \quad , \tag{7}$$

where $\Delta (\frac{x}{\ell})$ is the nondimensional distance traveled by the pressure peak during the time $\Delta\tau$. For wave shown in Figure 27, $\Delta (x/\ell) = 0.0139$ and $\Delta\tau = 0.628$, so that

$$\frac{\Delta (\frac{x}{\ell})}{\Delta \tau} = \frac{1}{\ell\omega}\, a = 0.0222 \tag{8}$$

From Equation (5), $\ell\omega = \frac{.972}{r}\, a_o$. Using this relation yields

$$\frac{a_o}{a} = .465 \tag{9}$$

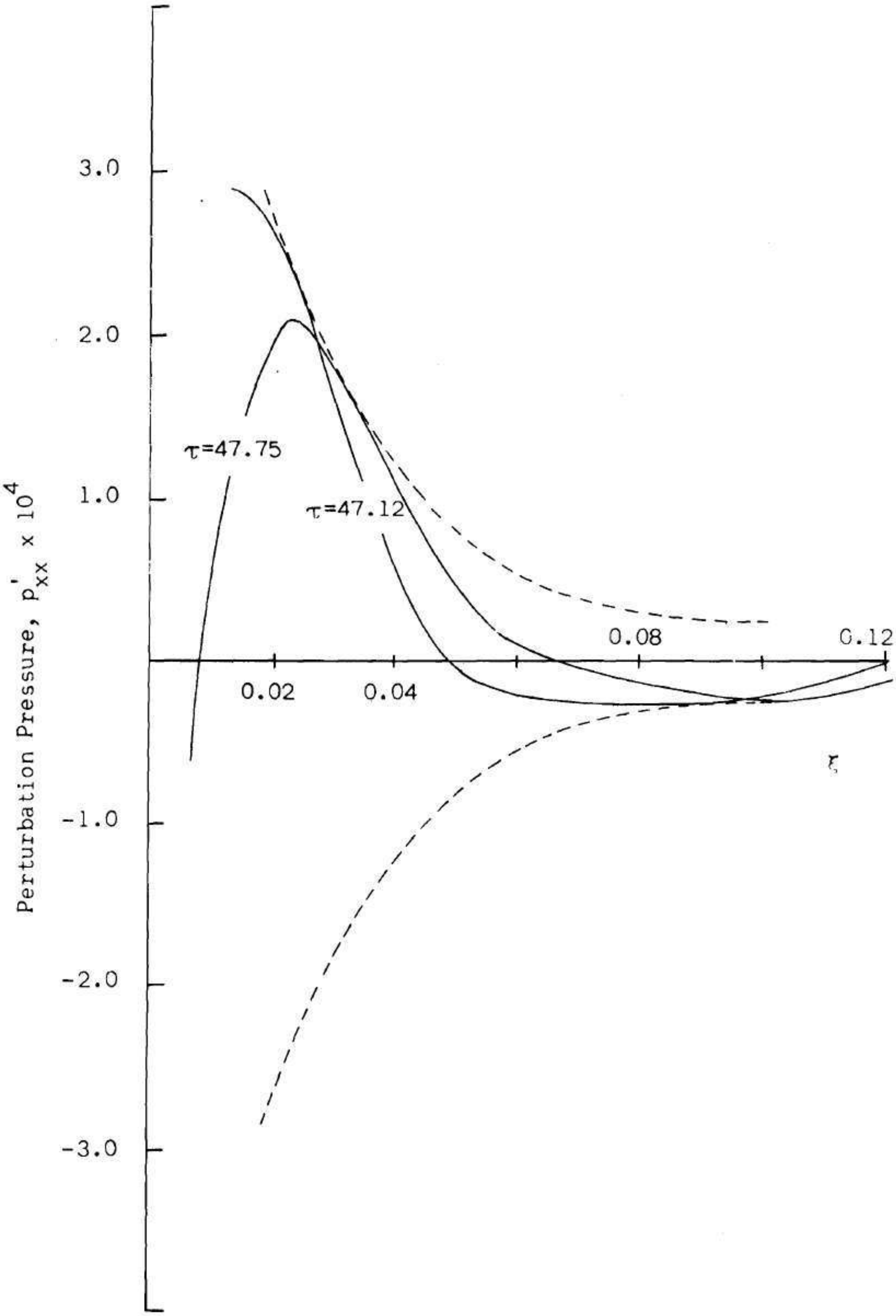

Figure 27. Steady State Perturbation Pressure Envelope and Two Typical Pressure Distributions.

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VITA

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